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Gauge Invariant Quark Propagator in the Instanton Background

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Abstract

After a general discussion on the choice of gauge, we compare the quark propagator in the background of one instanton in regular and singular gauge with a gauge invariant propagator obtained by inserting a path-ordered gluon exponential. Using a gauge motivated by this analysis, we were able to obtain a finite result for the quark condensate without introducing an infrared cutoff nor invoking some instanton model.

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1 Introduction

A variety of predictions concerning chiral symmetry breaking and concerning the lightest hadrons in various channels can be made within the instanton liquid model. Although there are various attempts to derive this model from first principles, it is still an open question, whether instantons melt or not. Thus the infrared problem remains unsolved.

In section 3, I present a small calculation of the quark condensate. I get a finite result by choosing an approriate gauge and performing a self energy resummation.

Because the finiteness essentially depends on the choice of gauge, I give a more general discussion in section 2 of how to choose a gauge when calculating gauge dependent quantities. The quark propagator in the background of one instanton in the well known regular and singular gauge is compared to a gauge invariant (GI) propagator, for which explicit expressions are calculated in this work.

The rest of section 1 contains a summary of well known instanton formulas, which are used in the subsequent sections.

Instantons in QCD

The solutions of the euclidian Yang-Mills equations fall into topology classes $N \in \mathbb{Z}$ and are called N instanton[2] solutions. The one instanton solution has the well-known form

$$\begin{aligned}
A^a_{I\mu}(x) &= O_I^{ab}\eta^{Q_I}_{b\mu\nu}\frac{(x-z_I)_\nu}{(x-z_I)^2}\frac{2\rho^2}{(x-z_I)^2+\rho^2} \quad \text{in singular gauge} \\
A^a_{I\mu}(x) &= O_I^{ab}\eta^{-Q_I}_{b\mu\nu}\frac{2(x-z_I)_\nu}{(x-z_I)^2+\rho^2} \quad \text{in regular gauge} \\
\gamma_I = (z_I, O_I, \rho_I, Q_I) &= (\text{location, orientation, radius, topological charge})
\end{aligned}$$

The parameters γ_I of the instanton simply reflect the symmetries of the Lagrangian (translation, rotation, scale invariance, parity). The partition function in semiclassical approximation is [1],[2]

$$\begin{aligned}
Z_1 &= \frac{1}{2}\sum_{Q_I=\pm1}\int d^4z_I dO_I d\rho_I\, D(\rho_I) \;=\; V_4\int_0^\infty d\rho\, D(\rho) \;=\; V_4\bar{D} \\
D(\rho) &= \frac{C_{N_c}}{\rho^5}S_0^{2N_c}e^{-S_1(\rho)} \;=\; C_{N_c}\rho^{-5}S_0^{2N_c}(\rho\Lambda)^b \\
C_{N_c} &= \frac{4.6e^{-1.679N_c}}{\pi^2(N_c-1)!(N_c-2)!} \\
S_0 &= \frac{8\pi^2}{g_0^2} \quad , \quad S_1(\rho) = \frac{8\pi^2}{g^2(\rho)} = b\ln\frac{1}{\rho\Lambda} \quad , \quad \Lambda = \Lambda_{PV} \\
S_2(\rho) &= \frac{8\pi^2}{g^2(\rho)} = b\ln\frac{1}{\rho\Lambda} + \frac{b'}{b}\ln\ln\frac{1}{\rho\Lambda} + O\left(\frac{1}{\ln\frac{1}{\rho\Lambda}}\right) \quad ,
\end{aligned}$$

$D(\rho)$ is the density of instantons of size ρ, $g(\rho)$ the running coupling constant, $b = \frac{11}{3}N_c$ and $b' = \frac{17}{3}N_c^2$. S_0 is the classical instanton action and g_0 is the unrenormalized tree level coupling constant. Whenever g_0 appears in some formulas, one has to guess its value. This unlucky situation may be improved by using the two loop expression for $D(\rho)$ replacing S_0 by S_1 and S_1 by S_2. But this is only an improvement for a small coupling. When ρ reaches the QCD scale Λ one should rely on a low order calculation to have a chance to get sensible results.

The Infrared Problem

The sum of widely separated instantons is also an approximate solution of the YM equations.

$$A = \sum_{I=1}^{N} A_I \quad , \quad S[A] \approx NS_0$$

[2] In this work instantons ($N > 0$) and anti-instantons ($N < 0$) will both be called instantons and are distinguished by their topological charge $Q = N$.

The partition function of this so called instanton gas is

$$Z = \sum_{N=0}^{\infty} Z_N \quad , \quad Z_N \approx \frac{1}{N!}(V_4 \bar{D})^N$$

The sum is dominated by an instanton density $N/V_4 = \bar{D}$. Unfortunately $\bar{D}$ is infinite and the assumption of a dilute instanton gas is inconsistent. This infinite density is caused by the divergence of $D(\rho)$ for large ρ, which in fact is a consequence of the increasing coupling constant at large distances.

There were several suggestions to overcome this problem. The most primitive is to introduce a cut-off ρ_c and ignore large instantons:

$$\bar{D}_{\rho_c} = \int_0^{\rho_c} d\rho \, D(\rho) \quad .$$

The cut-off is chosen small enough to make the spacetime fraction f filled with instantons less than 1 so that the dilute gas model is justified

$$f = \frac{2}{N_c} \int_0^{\rho_c} d\rho \, \frac{1}{2}\pi^2 \rho^4 D(\rho) \quad < \quad 1$$

This simple cut-off procedure can be improved by introducing a scale invariant hardcore repulsion between instantons, which effectively supresses large instantons [3]. This procedure has the advantage of respecting the scaling Ward identities which are otherwise violated by the simple cut-off ansatz. In [4] such an repulsion has been found leading to a phenomenologically welcomed packing fraction. Unfortunately this repulsion is an artefact of the sum-ansatz as has been shown by [7]. Therefore the infrared problem is still unsolved.

Nevertheless it is possible to make successful predictions by simply assuming a certain instanton density and some average radius. This instanton liquid model has been very successful in describing the physics of light hadrons [5],[6].

In high energy processes involving momenta p of $1 - 10$GeV, $D(\rho)$ is usually multiplied by a function sharply peaked at $\rho \sim p^{-1}$. The integral over ρ is now dominated by small instantons and infrared convergent. The results are therefore independent of the cut-off and no model has to be invented.

2 On the Choice of Gauge

Generalities

Gauge symmetry is a rather large symmetry, an infinite product of $SU(N_c)$ in the case of QCD. A physicist is always happy of having symmetries because they can be exploited to make predictions even without solving the theory. Gauge symmetry is neccessary to get a physical vector particle spectrum. As long as one does not make an approximation which manifestly breaks gauge symmetry one can choose a comfortable gauge for calculations because the result is GI. But it is very difficult *not* to break GI, especially in a non-abelian gauge theory. It is not easy to find a GI regularization and furthermore, the gluon propagator, the primary object in perturbation theory, is not GI. Of course it is meanwhile well known how to perform GI calculations in every order perturbation theory using FP-ghosts and dimensional regularization. Every new approach beyond perturbation theory is again confronted with the problem of GI. In lattice theory the Wilson action had to be invented. In Schwinger-Dyson and Bethe-Salpeter type selfconsistency equations GI is still an open problem. In instanton physics when going beyond the one instanton approximation the choice of gauge is also important. This will be discussed in the next paragraph. There is a related problem when considering non-GI objects from the very beginning like the gluon or quark propagator. Strictly speaking they are only *defined* when relying to a certain gauge. In principle one should not give them any physical meaning at all. Often one is tempted to do so and therefore it is necessary to give some motivation of choosing this or that gauge.

A Natural Gauge

The gauge field A_μ^a describes the connection between neighbouring vector bundles over the spacetime manifold $\mathbb{R}^4$. Thus a choice of gauge is like the choice of a coordinate system in general relativity with connection $\Gamma^\mu{}_{\nu\rho}$. When choosing a crooked coordinate system, although being in a smooth universe, there will appear fictitious accelerations towering above the real physical accelerations

$$\ddot{x}^\mu_{phys} = \ddot{x}^\mu_{fict} + \Gamma^\mu{}_{\nu\rho}\dot{x}^\nu\dot{x}^\rho \quad .$$

When making general covariant calculations these fictitious accelerations and the Γ contribution will cancel out thus leading to the correct small result. But the slightest unsystematic approximation will produce gross errors. The natural solution of this problem is to use a coordinate system as smooth as possible to avoid fictitious accelerations, e.g. to choose $\Gamma^\mu{}_{\nu\rho}$ as small as possible. To make this statement more quantitative we may try to minimize $(\ddot{x}^\mu_{phys} - \ddot{x}^\mu_{fict})^2$ simultaneously for all curves. This is done by choosing a coordinate system which minimizes[3]

$$||\Gamma||^2 \;:=\; \int \Gamma_\mu{}^{\nu\rho}\Gamma^\mu{}_{\nu\rho}\, d^4x$$

This obviously measures the crookedness of the coordinate system.

Let us now transfere this to QCD. The analog norm for the gauge potential is

$$||A||^2 \;:=\; \int tr_c A_\mu A^\mu\, d^4x$$

A stationary point is found by variating $||A||$ w.r.t. gauge transformations

$$\delta A_\mu = i[A_\mu,\Omega] + \partial_\mu\Omega \quad , \quad \delta||A||^2 = 2i\int tr(\partial_\mu A^\mu)\Omega d^4x = 0 \quad \forall\Omega \quad \Longleftrightarrow \quad \partial_\mu A^\mu = 0$$

[3]In Euclidian space this is a positive definite norm

Therefore in Lorentz gauge, A_μ^a contains as few pure gauge as possible, if the stationary point is a minimum. An expansion in A is thus most rapidly convergent in Lorentz gauge. In applications where A is not needed in total e.g. when only a certain momentum region is probed, different norms and different gauges may be optimal in the sense discussed above. Especially one should include derivatives of A into the norm in order to guarantee a smooth A which is important for high energies.

On the Gauge in Instanton Physics

When calculating GI quantities in the background of one instanton in a GI way the choice of gauge is only a matter of convenience. But one can see that there are large cancelations between different terms in regular gauge at large distances due to their slow decay and in singular gauge at small distances due to the topological singularity at the instanton center. For non-GI invariant quantities like the gluon or quark propagator, or when making some unsystematic approximation, the lesson is to use singular/regular gauge when dealing with low/high energies to avoid these cancelations. This is consistent with the discussion given above. Singular as well as regular gauge fulfill the Lorentz condition. $||A_{sing}||$ is finite and a minimum. A_{sing} is therefore a good choice for low energies. For high energies it is important to have a smooth A which is obviously only satisfied by the regular gauge.

To linearly superpone instantons they have to decay rapidly enough. Therefore one has to use singular gauge. This argument can in principle be circumvented by superponing two fields A_N and $A_{\overline{N}}$ the former/latter being an exact multi-instanton/anti-instanton configuration in regular gauge. Despite this, for low energies singular gauge is in any case a good choice and for high energies a one instanton approximation is already a good approximation.

The Quark Propagator in Axial Gauge

A specific example to test the gauge dependence is the quark propagator. The contribution of one instanton of radius ρ to $M(p) = ip^2\bar{S}_I(p)$ which usually is interpreted as a constituent quark mass is shown in figure 1 in regular, singular and axial gauge. The regular graph is larger than the singular at low momentum and the singular graph shows the slow decay (only polynomial in $1/p$) for large momenta. The analytical expressions are well known and are listed in appendix A together with expressions in axial gauge which will be derived and discussed below.

A correlator containing color-non-singlet operators can be made GI by connecting distant points with a special path-ordered exponential containing the gauge field. The exponential ensures the parallel transport of color from one point to the other. The GI quark propagator may symbolically written as

$$S_{ax}(x,y) = \langle 0|\Psi(x) P \exp\left(i\int_x^y dz\cdot A(z)\right)\bar{\Psi}(y)|0\rangle \tag{1}$$

P denotes path-ordering. We have already defined S_{ax} to be its color singlet part because only the singlet part is GI. S_{ax} will be called the axial propagator because in axial gauge with $n_\mu = x_\mu - y_\mu$ the exponential vanishes. In the one instanton background in zeromode approximation we get

$$S_{ax}(x,y) = \frac{\mathbb{1}_c}{N_c} tr_c \left[P \exp\left(i \int_x^y dz \cdot A(z) \right) \psi(x)\bar{\psi}(y) \right]$$

where A is now the instanton field and ψ is the zeromode in any gauge. In a coordinate system where the instanton sits at the origin and $x-y$ is in time direction[4] ($\mathbf{x}=\mathbf{y}=\mathbf{z}$) the path ordered exponential reduces to an ordinary exponential. Alternatively we could have tried to find a gauge transformation which transforms the regular gauge in axial gauge. In both cases we get:

$$\begin{aligned} S_{ax}(x,y) &= \frac{1}{N_c} tr_c \left[\psi_{ax}(x) \bar{\psi}_{ax}(y) \right] \quad , \quad \psi_{ax}(x) = R(x)\psi_{reg}(x), &(2)\\ R(x) &= e^{\pm i\alpha(x)\frac{\tau\mathbf{x}}{|\mathbf{x}|}} = \cos\alpha(x) \pm i\frac{\tau\cdot\mathbf{x}}{|\mathbf{x}|}\sin\alpha(x) =: \pm i\tau_\mu^\pm \cdot \tilde{x}(x) &(3)\\ \alpha(x) &= \frac{|\mathbf{x}|}{\sqrt{\mathbf{x}^2+\rho^2}} \arctan \frac{x_0}{\sqrt{\mathbf{x}^2+\rho^2}} &(4)\end{aligned}$$

$\alpha(x)$ may also be written in a covariant form

$$\alpha(x) = \pm \left(1 + \frac{\rho^2(x-y)^2}{x^2y^2-(xy)^2} \right)^{-1/2} \arctan \sqrt{\frac{(x^2-(xy))^2}{x^2y^2-(xy)^2+\rho^2(x-y)^2}}$$

but now $\alpha(x)$ depends also on y and the expression for the propagator no longer factorizes. The reason for this is that the axial gauge is not covariant, but the definition of the propagator is. Inserting (3) and (12) into (2) we get

$$S_{ax}(x,y) = \frac{1}{N_c} \left((\tilde{x}\tilde{y}) - \frac{1}{2}\tilde{x}_\mu \tilde{y}_\nu \sigma^{\mu\nu} \right) \frac{1\pm\gamma_5}{2} \varphi_{reg}(x)\varphi_{reg}(y) \tag{5}$$

Inserting (3), (4) and (12) into (5) the space-time averaged propagator can be expressed as an integral over elementary functions

$$\bar{S}(x-y) = \int_0^\infty dr \int_{-\infty}^\infty dt \, 4\pi r^2 \cos\left[\frac{r}{R} \left(\arctan\frac{t+|x-y|}{R} - \arctan\frac{t}{R} \right) \right] \cdot$$

$$\cdot \frac{1}{2N_c} \frac{\rho^2}{\pi^2(R^2+(t+|x-y|)^2)^{3/2}(R^2+t^2)^{3/2}} \quad , \quad R^2 = r^2+\rho^2$$

The difference between the propagator in regular and axial gauge is the insertion of the $\cos[\ldots]$ factor. Therefore the axial propagator is everywhere smaller than the regular propagator, except at $x=y$ where they coincide because the path-ordered exponential

[4] Although working in Euclidian space we will adopt the Minkowskian language $(x_0, \mathbf{x})$ = (time, space).

is one. At large distances it is smaller by a factor $\pi/4$. Instead of performing the integration in coordinate space, let us go directly to the more interesting momentum space representation:

$$\bar{S}_I(p) = \frac{1}{2N_c}\varphi^\mu_{ax}(p)\varphi^{\mu\dagger}_{ax}(p) \quad , \quad \varphi^\mu_{ax}(p) = \int \tilde{x}^\mu(x)\varphi_{reg}(x)e^{ipx}dx$$

Although φ^μ_{ax} does not transform like a vector, we can choose a convenient direction of p because $\varphi\varphi^\dagger$ is a Lorentz scalar. For pure spacelike p the spacial components of φ vanish because the integrand is anti-symmetric w.r.t time reflection. Only the time component is nontrivial

$$\varphi^0_{ax}(p) = \int d^3r \int dt \, \cos\left[\frac{r}{R}\arctan\frac{t}{R}\right]\frac{\rho}{\pi(R^2+t^2)^{3/2}}e^{i\mathbf{p}\cdot\mathbf{r}}$$

With the following hints

$$\begin{aligned}
\cos(\gamma\arctan x) &= \mathrm{Re}\left(\frac{1+ix}{1-ix}\right)^{\gamma/2} \\
\int_{-\infty}^{\infty}(R-it)^{-\alpha}(R+it)^{-\beta}dt &= 2\pi(2R)^{1-\alpha-\beta}\frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)\Gamma(\beta)} \\
\Gamma(\frac{3}{2}-x)\Gamma(\frac{3}{2}+x) &= \frac{(1/4-x^2)\pi}{\cos\pi x} \\
\int d^3r\, e^{i\mathbf{p}\cdot\mathbf{r}}f(r) &= \frac{2\pi}{p}\int_0^\infty f(r)\sin(pr)r\,dr
\end{aligned} \tag{6}$$

the reader should be able to perform the t and the angular integration $d\Omega_r$,

$$\varphi^0_{ax}(p) = \frac{8}{p\rho}\int_0^\infty \cos\left(\frac{\pi r}{2R}\right)\sin(pr)r\,dr$$

I was not able to perform this last integral analytically, but for small momenta it is easy to see that $\varphi^0_{ax}(p)$ behaves like $\pi^2\rho/p$. For large p it decays like $\sim e^{-p\rho}$ with a non-polynomial coefficient because of an essential singularity at $r=\pm i\rho$. Comparing φ_{reg}, φ_{sing} and φ_{ax} plotted in figure 1 we see that the axial φ lies somewhat in between the regular and the singular. So one may conclude that axial gauge is a good compromise for all momenta.

The calculation of the GI propagator seems to make the discussion of its gauge dependent partners obsolet. I will now argue that this is not the case. The reason is that there are a huge number of GI definitions of a quark propagator and (1) is only one possible choice. One obvious generalization is to choose a more complicated path from x to y than a straight line. The next thing one could do is not to restrict oneself to a specific path, but to take into account all paths one is interested in and average the results with arbitrary weights. Another possibility is to let the path depend on the gauge field itself, as long as this choice is made in a GI way. Finally one can combine both generalizations. I am sure that it is possible to produce any result for the propagator with a suitable generalized definition. The advantage of the standard axial propagator is, that the definition is simple and that the non local operator has a physical interpretation. It creates a quark-antiquark pair connected by a thin gluon flux tube. This might be a good choice for a non-local meson creation operator. But it is also plausible that one of the generalizations given above is even better. The only thing I want to point out is, that the GI definition for the propagator given above is nothing more than to work in axial gauge. One still has to choose the right gauge using more sophisticated arguments.

3 The Quark Condensate

$N_c \to \infty$

The only reason for performing the $N_c \to \infty$ limit is to make the N_c dependence of the resulting formulas simple. The accuracy has been checked to be within the standard 10% for $N_c = 3$ usually achieved by $1/N_c$ expansion. Here the accuracy can simply be understood. The actual expansion parameter is not $1/N_c = 1/3$ itself but $1/b \approx 1/11$. The following asymptotic formulas will be used

$$N_c!^{1/N_c} \doteq N_c/e \quad , \quad b \doteq \frac{11}{3}N_c \quad , \quad C_{N_c}^{1/b} \doteq 2.22 b^{-6/11}$$
$$\rho^5 D(\rho) \sim (2.22(S_0/b)^{6/11}\rho\Lambda)^b \quad , \quad S_0/b = \frac{24\pi^2}{11}(g_0^2 N_c)^{-1} \tag{7}$$

Every equality in the large N_c limit will be marked with a dot. Notice that in this limit instantons of size $\rho < \frac{1}{2.22}(S_0/b)^{-6/11}\Lambda^{-1}$ are completely suppressed. Above this threshold the instanton density gets infinite. S_0/b is independent of N_c because $g_0 \sim 1/\sqrt{N_c}$.

Effective Quark Mass

In the presense of one light quark flavor the instanton density $D(\rho)$ has to be multiplied with the functional determinant of the Dirac operator

$$Det(i\not{D} + im) \approx 1.34 m\rho$$

which is proportional to m because of a zeromode of $\not{D}$. The quark propagator in the background of one instanton is dominated by this zeromode

$$S_I(p,q) = \frac{\psi_I(p)\psi_I^\dagger(q)}{im}$$

Averaging this expression over all collective coordinates γ_I one gets

$$M(p) := ip^2\bar{S}_I(p) = \frac{1.34}{2N_c}\int_0^\infty d\rho\, p^2 \rho D(\rho)\varphi^2(p)$$

Summing the contribution to the propagator of $0,1,2,3,\ldots$ instantons, which is the analog of a selfenergy resummation in perturbation theory,

$$S(p) = \frac{1}{\not{p}} + \frac{1}{\not{p}}\frac{M(p)}{i}\frac{1}{\not{p}} + \frac{1}{\not{p}}\frac{M(p)}{i}\frac{1}{\not{p}}\frac{M(p)}{i}\frac{1}{\not{p}} + \ldots = \frac{1}{\not{p} + iM(p)}$$

justifies to call $M(p)$ a dynamical quark mass. Expressions of φ in various gauges are given in appendix A. The graphs of $\varphi(p)$ in singular and regular gauge cross over at

$$\varphi_{sing}(p) = \varphi_{reg}(p) \quad \Longleftrightarrow \quad 2p\rho \approx 2.5$$

Therefore one should use regular gauge for large ρ and singular gauge for small ρ. This choice of gauge also makes the integral convergent for large ρ. At this stage we have no infrared problem. Using regular gauge in the whole integration interval we get

$$M_{reg}(p) = Bp(\frac{\Lambda}{2p})^b \quad , \quad B = 1.34 \cdot 16\pi^2 (C_{N_c} b^{2N_c})(S_0/b)^{6b/11} I_b/N_c$$

$$I_b = \int_0^\infty dz\, z^{b-2} e^{-z} = (b-2)! \quad , \quad z = 2p\rho$$

The integral is sharply dominated by

$$z \doteq b \pm b^{1/2} >> 2.5$$

therefore the result is independent of the choice of gauge for $z < 2.5$ justifying our use of regular gauge over the whole integration interval. Axial gauge would lead to nearly the same result as can be seen from figure 1. Using singular gauge for large ρ would produce a divergent integral dominated by arbitrary large instantons inconsistent with the choice of gauge discussed above. The infrared "problem" shows up in the rapid raise of $M(p)$ for low p, which effectively supresses the propagation of quarks with low virtuality p^2. Consider some process involving quarks at distances $x = 1/p_c$. The effective quarkmass $M(1/x)$ is dominated by instantons of much larger size

$$\rho = \rho_c(1 \pm b^{-1/2}) >> x \quad , \quad \rho_c = \frac{b}{2p_c}.$$

In other words, given an instanton of radius ρ influences the physics at a much smaller scale $x = \frac{2}{b}\rho << \rho$. Therefore the interior of the instanton is probed and one should avoid the singularity at its center by using regular gauge.

The Quark Condensate

Let us now calculate a real physical gauge invariant observable, the quark condensate

$$\langle\bar\psi\psi\rangle := \lim_{x\to 0} \mathrm{tr}_{CD}(S(x) - S_0(x)) = -4iN_c \int \frac{M(p)}{p^2 + M^2(p)} \frac{d^4p}{(2\pi)^4}$$

Inserting M(p) and performing the angular integration we get

$$|\langle\bar\psi\psi\rangle| = \frac{N_c}{16\pi^2} B^{3/b} J_b \Lambda^3$$

$$J_b = \int_0^\infty \frac{z^{b+2}}{1 + z^{2b}} dz = \frac{\pi}{2b \sin(\frac{b+3}{2b}\pi)} \doteq \frac{\pi}{2b} \quad , \quad p = B^{1/b} \frac{\Lambda}{2} z$$

The integral is finite and sharply dominated by $z \doteq 1 \pm b^{-1}$. Without resummation of the selfenergies the integral J_b and thus condensate would have turned out to be infinite. The condensate is dominated by quark wavefunctions with momenta

$$p = p_c(1 \pm b^{-1}) \quad , \quad p_c = \beta b \frac{\Lambda}{2} \quad , \quad \beta := \frac{1}{b} B^{1/b} \doteq \frac{2.22}{e}(S_0/b)^{6/11}$$

and depends on Λ and g_0

$$|\langle \bar{\psi}\psi \rangle|^{1/3} = 0.139 \beta b \Lambda \quad .$$

Discussion

Expressing p_c and ρ_c in terms of $|\langle \bar{\psi}\psi \rangle|$ by eliminating β we get our main result

$$\begin{aligned} p_c &= 3.59 |\langle \bar{\psi}\psi \rangle|^{1/3} \\ \rho_c &= \frac{1.96}{N_c} |\langle \bar{\psi}\psi \rangle|^{1/3} && (8) \\ 2.22 N_c \Lambda &= (g_0^2 N_c)^{6/11} |\langle \bar{\psi}\psi \rangle|^{1/3} && (9) \end{aligned}$$

A weak point is the experimental extraction of g_0. It should be extracted from a reliable treelevel process at low energies presumably of the order of ρ_c. In QCD improved Bag-Models the main nonperturbative effect is modeled by the bag and the hyperfinesplitting is caused by a one gluon exchange. g_0 extracted from $\Delta - N$ splitting is [8]

$$g_0^{bag} \approx 2.6$$

Let me also give a theoretical guess of g_0. The change to a two loop expression for the instanton density $S_{0/1} \rightsquigarrow S_{1/2}$ can be effectively performed by only replacing S_0 in the following way

$$(S_0/b)^{6/11} \rightsquigarrow (\ln \frac{1}{\rho\Lambda})^{\alpha} \quad , \quad \alpha = \frac{15}{121}$$

Because α is very small $(\ln \frac{1}{\rho\Lambda})^{\alpha}$ is approximately one in a large range of values for $\rho\Lambda$. and for

$$g_0^{guess} = 2.7\sqrt{3/N_c}$$

the two 2 loop density coincides with the one loop density. Because we do not believe that the 2 loop density is an improvement, one should not take g_0^{guess} too seriously. At least it is not in contradiction with g_0^{bag}.

The condensate is well known to be $|\langle \bar{\psi}\psi \rangle|^{1/3} = 240\text{MeV}$. Setting $N_c = 3$ and taking $g_0 = 2.6$ for granted we get

$$\begin{aligned} p_c \pm \Delta p &= (860 \pm 80)\text{MeV} \\ \rho_c \pm \Delta\rho &= (160 \pm 50)^{-1}\text{MeV} && (10) \\ \Lambda_{PV} &\approx 190\text{MeV} \end{aligned}$$

The most interesting thing is, that the condensate is sharply dominated by quark field wave functions of rahter large momentum p_c. On the other hand the dominating instantons have a very large radius ρ_c, 4 times larger than usually assumed in instanton liquid models. Nevertheless the predicted value of Λ_{PV}, which of course must be assigned a large error because of the rough estimate of g_0, is in agreement with experiment.

4 Conclusion

Whenever one is calculating gauge dependent objects or when making gauge breaking approximations, one is confronted with the problem of choosing a "good" gauge. Specializing the general discussion of section 2 to the case of instantons, we came to the conclusion that the regular gauge is appropriate for small distances and the singular gauge for processes involving large distances. The GI propagator was defined, calculated and compared to the propagator in singular and regular gauge (figure 1). The conclusion was, that the GI propagator is not a-priori a good choice, but lies somewhat in between regular and singular gauge.

Using an appropriate gauge along the lines discussed in section 2 we were able to derive a finite quark condensate without taking an infrared cut-off for the instanton radius nor relying on some instanton model. The linear relation between $|\langle\bar\psi\psi\rangle|^{1/3}$ and the QCD scale Λ is in agreement with experiment. The condensate if formed by quark fields of high momenta $p_c = 860\mathrm{MeV}$ mainly lying within the sharp region $\Delta p = 80\mathrm{MeV}$. The dominating instantons are very large ($\rho_c = 160\mathrm{MeV}$).

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A Instantons in Singular, Regular and Axial Gauge

The instanton at the origin in standard orientation is given in singular, regular and axial gauge:

$$\begin{aligned} A_\mu^{sing}(x) &= \eta^\pm_{\mu\nu}\frac{x_\nu}{x^2}\frac{\rho^2}{x^2+\rho^2} \quad , \quad \tau^\pm_\mu\tau^\mp_\nu = \delta_{\mu\nu} + i\eta^\pm_{\mu\nu} \\ A_\mu^{reg}(x) &= \eta^\mp_{\mu\nu}\frac{x_\nu}{x^2+\rho^2} \quad , \quad \tau^\pm_\mu = (\pm i, \tau) \\ A_\mu^{ax}(x) &= R(x)A_\mu^{reg}(x)R^\dagger(x) + iR(x)\partial_\mu R^\dagger(x) \end{aligned} \tag{11}$$

The upper/lower sign corresponds to an instanton/anti-instanton ($Q = \pm 1$).

$$R(x) = \pm i\tau^\pm_\mu \tilde{x}^\mu(x) \quad , \quad \tilde{x}^\mu(x) = \begin{pmatrix} \cos\alpha(x) \\ \frac{\mathbf{x}}{|\mathbf{x}|}\sin\alpha(x) \end{pmatrix}$$

$$\alpha(x) = \frac{|\mathbf{x}|}{\sqrt{\mathbf{x}^2+\rho^2}}\arctan\frac{x_0}{\sqrt{\mathbf{x}^2+\rho^2}}$$

The covariant derivative $\not{D}$ has one zeromode

$$i\not{D}\psi = (i\not{\partial} - \not{A})\psi = 0$$

where the zeromode has the following form:

$$\begin{aligned}
\psi_{sing}(x) &= \sqrt{2}\varphi_{sing}(x)\not{x}\chi \quad , \quad \varphi_{sing}(x) = \frac{\rho}{\pi|x|(x^2+\rho^2)^{3/2}} \\
\psi_{reg}(x) &= \sqrt{2}\varphi_{reg}(x)\chi \quad , \quad \varphi_{reg}(x) = \frac{\rho}{\pi(x^2+\rho^2)^{3/2}} \\
\psi_{ax}(x) &= \sqrt{2}\varphi_{reg}R(x)\chi
\end{aligned} \tag{12}$$

χ is a color Dirac spinor given by

$$\chi^{\pm}\bar{\chi}^{\pm} = \frac{1}{16}\gamma_\mu\gamma_\nu\frac{1\pm\gamma_5}{2}\tau_\mu^{\mp}\tau_\nu^{\pm}$$

For light quarks the propagator is dominated by the zeromode. When averaged over the instanton orientation, position and charge the propagator is diagonal in momentum space and given by

$$\langle\psi(p)\psi^\dagger(p)\rangle = \frac{1}{2N_c}\varphi^2(p)$$

where

$$\begin{aligned}
\varphi_{sing}(p) &= \pi\rho^2\frac{d}{dz}[I_1(z)K_1(z) - I_0(z)K_0(z)]_{z=p\rho/2} \\
\varphi_{reg}(p) &= \frac{4\pi\rho}{p}e^{-p\rho} \\
\varphi_{ax}(p) &= \frac{8}{p\rho}\int_0^\infty \cos\left(\frac{\pi r}{2\sqrt{r^2+\rho^2}}\right)\sin(pr)r\,dr
\end{aligned} \tag{13}$$

The asymptotics are given in the following table

$\frac{p}{\rho}\varphi(p)$	singular	regular	axial
$p\rho \ll 1$	2π	4π	π^2
$p\rho \gg 1$	$\frac{12\pi}{(p\rho)^3}$	$4\pi e^{-p\rho}$	$\sim e^{-p\rho}$

Table 1: *Asymptotic behaviour of* $\frac{p}{\rho}\varphi(p)$

The constituent mass of a quark in the gas approximation is

$$M(p) = ip^2\bar{S}(p) = 1.34\int_0^\infty d\rho\,\rho D(\rho)\langle\psi(p)\psi^\dagger(p)\rangle$$

Only when the instanton radius is kept fixed, the mass is proportional to

$$M(p) \sim p^2\varphi^2(p) \quad .$$

$p^2\varphi^2(p)$ is plotted in figure 1 in all three gauges.

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C Figures

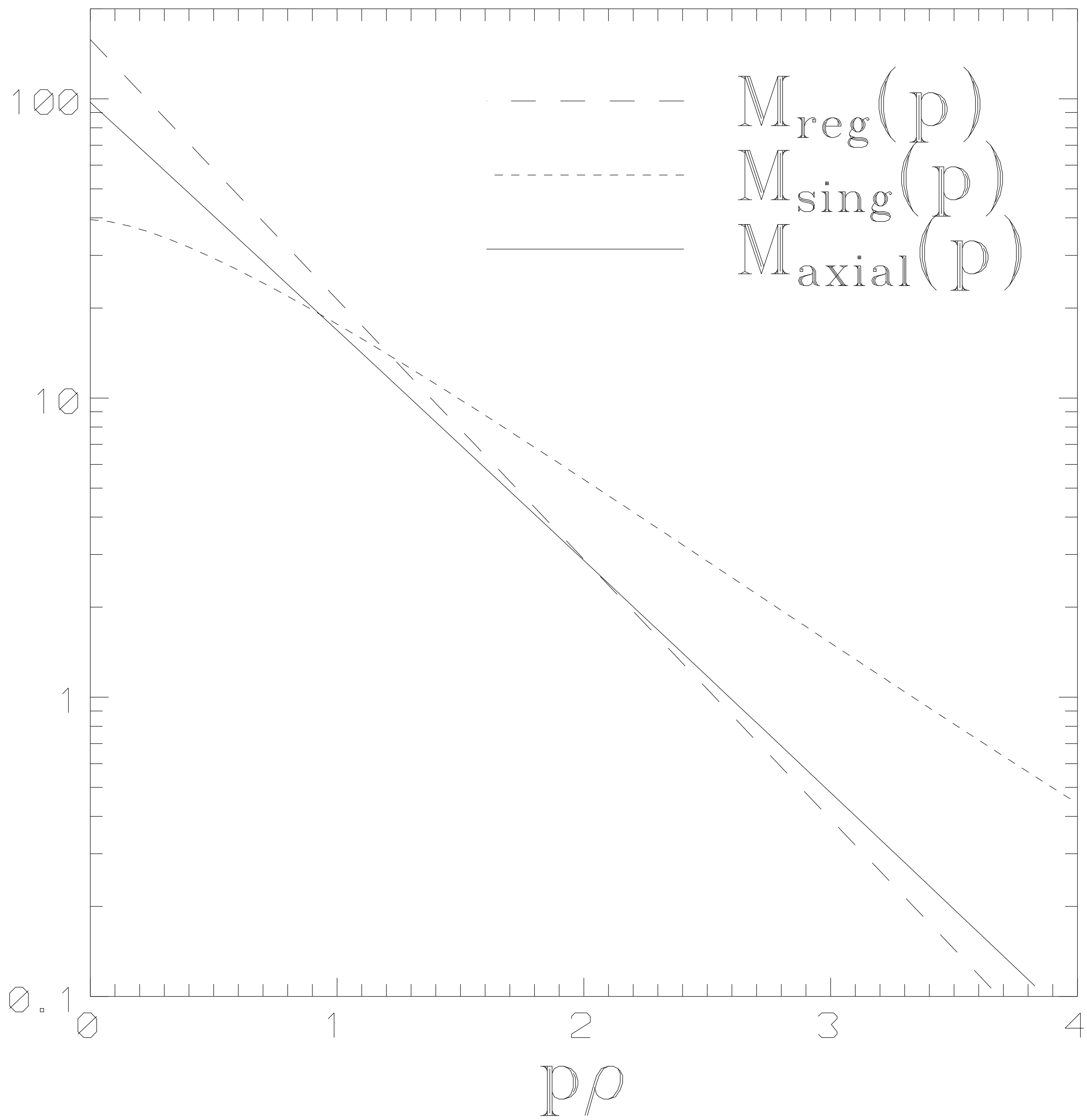

Figure 1: *Constituent quark mass* $M(p) \sim p^2\varphi^2(p)$ *in singular, regular and axial gauge for fixed instanton radius* ρ *in arbitrary normalization. For a given momentum the corresponding lowest curve may be interpreted as the "most physical" one.*