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The Mass of the η' in selfdual QCD

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Abstract

The QCD gauge field is modeled as an ensemble of statistically independent selfdual and antiselfdual regions. This model is motivated from instanton physics. The scale anomaly then allows to relate the topological susceptibility to the gluon condensate. With the help of Wittens formula for $m_{\eta'}$ and an estimate of the suppression of the gluon condensate due to light quarks the mass of the η' can be related to f_π and the physical gluon condensate. We get the quite satisfactory value $m_{\eta'} = 884 \pm 116\,\text{MeV}$. Using the physical η' mass as an input it is in princple possible to get information about the interaction between instantons and anti-instantons.

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1 Introduction

In many channels a direct calculation of the meson correlators in the instanton liquid model and a spectral fit lead to reasonable results for the masses of the lightest mesons [1]. This method even works in the axial triplet channel because the model correctly describes spontaneous breaking of chiral symmetry. In the axial singlet channel a strong repulsion prevents the formation of a meson [2, 9]. The conclusion is, that there is no massless Goldstone boson in this channel, but the mass of the η' remains undetermined. In this letter I want to calculate the mass of the η' by combining quite different techniques. With the help of

- current algebra theorems for the η',
- $1/N_c$ expansion,
- instanton model,
- scale anomaly

we are able to relate the η' mass to the pion coupling constant f_π and the physical gluon condensate.

2 Wittens formula

In leading order in $1/N_c$ it is possible to relate the η' mass to the Θ dependence of the topological susceptibility[2] $d^2E/d\Theta^2$ of QCD without quarks [3]:

$$m_{\eta'}^2 = \frac{4N_f}{f_\pi^2}\left(\frac{d^2E}{d\Theta^2}\right)_{\Theta=0}^{no\ quarks} \quad , \quad \frac{d^2E}{d\Theta^2} = \int d^4x\ \langle 0|\mathcal{T}Q(x)Q(0)|0\rangle_{conn} \tag{1}$$

$$Q(x) = \frac{\alpha_s}{4\pi}\mathrm{tr}_c G\tilde{G}(x) \quad , \quad Q = \int d^4x\, Q(x) \in \mathbb{Z}$$

$Q(x)$ is the topological charge density and Q the total charge. This formula is derived by arguing, that for large N_c the topological susceptibility is dominated by the η' state, utilizing the axial anomaly [8] and the relation $f_\pi = f_{\eta'}$, which is exact for $N_c \to \infty$.

3 Selfdual QCD

The next step is to relate the topological susceptibility to the gluon condensate $\langle 0|N(0)|0\rangle$:

$$N(x) = \frac{\alpha_s}{4\pi}\mathrm{tr}_c GG(x) \quad , \quad N = \int d^4x\, N(x)$$

[2] $\langle AB\rangle_{conn} = \langle AB\rangle - \langle A\rangle\langle B\rangle$

In instanton models the gluon field consists of instantons of charge $Q=\pm 1$. The exact N instanton solutions ($Q=N$) are selfdual ($G_{\mu\nu}=\tilde{G}_{\mu\nu}$) and

$$\langle 0|\mathcal{T}Q(x)Q(0)|0\rangle_{conn}=\langle 0|\mathcal{T}N(x)N(0)|0\rangle_{conn} \quad . \tag{2}$$

The exact anti-instanton solutions ($Q=-N$) are anti-selfdual ($G_{\mu\nu}=-\tilde{G}_{\mu\nu}$) and (2) holds too, because the two minus signs cancel. Unfortunately these exact solutions are not the most important contributions to the partition function.

The dominating configurations are instantons and anti-instantons in mixed combination. The simplest model is a dilute sum $A=\sum_I A_I$ of instantons of mixed charge. $G_{\mu\nu}$ is then approximately selfdual near the instanton centers, approximately anti-selfdual near the anti-instanton centers and small far away from any instanton. In leading order in the instanton density we have

$$G_{\mu\nu}(x)=\pm\tilde{G}_{\mu\nu}(x) \tag{3}$$

where the sign now depends on x! Let us define $N^{\pm}(x)$ in the following way:

$$N(x)=N_+(x)+N_-(x) \quad , \quad Q(x)=N_+(x)-N_-(x)$$

$N_+(x)$ is a sum of bumps near the centers of instantons and $N_-(x)$ near the centers of anti-instantons. (2) has to be replaced by the relation

$$\langle 0|\mathcal{T}Q(x)Q(0)|0\rangle_{conn}=\langle 0|\mathcal{T}N(x)N(0)|0\rangle_{conn}-4\langle 0|\mathcal{T}N_+(x)N_-(0)|0\rangle_{conn} \tag{4}$$

Assuming independence of instantons and anti-instantons ($\langle N_+N_-\rangle=\langle N_+\rangle\langle N_-\rangle$) the equation reduces again to (2). We will see that this is a crucial assumption.

4 The Scale Anomaly

The next ingredient is the scaling behaviour of QCD. Classical chromodynamics is scale invariant and the Noether theorem leads to a conserved scale current. In quantum theory the scale invariance is anomalously broken (like the axial singlet current). Ward identities can be derived, especially [6]

$$\int d^4x\ \langle 0|\mathcal{T}N(x)N(0)|0\rangle_{conn}=\frac{4}{b}\langle 0|N(0)|0\rangle \quad , \quad b=\frac{11}{3}N_c \tag{5}$$

Therefore in a self(anti)dual background the topological susceptibility $d^2E/d\Theta^2$ is proportional to the gluon condensate:

$$\frac{d^2E}{d\Theta^2}=\frac{4}{b}\langle 0|N(0)|0\rangle \tag{6}$$

The relation is still valid, when there are statistically independent regions of selfduality and selfantiduality, as discussed above. It is in fact sufficient to assure independence of the total instanton/anti-instanton number $N_\pm$. I have checked (6) by using the theoretical one loop instanton density $D(\rho)$ calculated in [10]. Only the ρ dependence $D(\rho)\sim\rho^{-5}(\rho\Lambda)^b$ is important. Due to the infrared divergence it is neccessary to introduce an infrared cutoff, but one has to assure not to break scale invariance. A minimal change is to introduce two cutoffs $f_\pm$ in the total instanton/anti-instanton packing fraction. The packing fraction is the spacetime volume ocupied by the instantons and is a dimensionless quantity. Scale invariance and independence of instantons and anti-instantons are ensured. The partition function Z is

$$Z = \sum_{N_+ N_-} Z^+_{N_+} Z^-_{N_-} \tag{7}$$

$$Z^\pm_{N_\pm} = \frac{V_4^{N_\pm}}{N_\pm !} \int_0^\infty d\rho_1 \dots d\rho_{N_\pm}\, D(\rho_1)\dots D(\rho_{N_\pm})\, \Theta\left(f_\pm - \frac{1}{V_4} \sum_{i=1}^{N_\pm} \rho_i^4 \right)$$

A lengthy, but quite standard calculation of statistical physics, leads to

$$Z^\pm_{N_\pm} = \left(\frac{c_\pm N_\pm}{V_4 \Lambda^4} \right)^{-\frac{bN_\pm}{4}}$$

where $c_\pm = c_\pm(f_\pm, b)$ are constants independent of $N_\pm$. Differentiation of $\ln Z$ w.r.t. $\ln c_\pm$ two times leads to (6). The result is independent of $f_\pm$. An attractive interaction would lower the susceptibility (compared to the density). This can be seen in the following way: In the extrem case of a very attractive interaction, all instantons will be bound to intanton-anti-instanton molecules, thus $N_+ = N_-$ and $Q \equiv 0$. On the other hand a repulsive interaction would increase the susceptibility:

$$\frac{d^2 E}{d\Theta^2} < \frac{4}{b} \langle 0|N(0)|0\rangle \quad \text{for attractive } I\overline{I} \text{ interaction} \tag{8}$$

$$\frac{d^2 E}{d\Theta^2} > \frac{4}{b} \langle 0|N(0)|0\rangle \quad \text{for repulsive } I\overline{I} \text{ interaction} \tag{9}$$

Therefore the violation of (6) is a measure of the $I\overline{I}$ interaction.

All this should be compared to Dyakonov [4], where the relation

$$d^2 E / d\Theta^2 = \langle 0|N(0)|0\rangle \tag{10}$$

has been derived, which is similar to (6). In this work a simple sum ansatz $A = \sum A_I$ has been made. This ansatz leads to a strong repulsion between close instantons, which is the origin of the missing factor $4/b$. The result on its own and the comparison with (9) shows that some repulsive interaction is at work. Verbaarshot [5] has shown that this repulsion is an artefact of the simple sum ansatz. Using the much more accurate and elaborate streamline ansatz he showed, that the interaction strongly depends on the orientation and the average interaction is about 14 times smaller than those obtained in [4]. Therefore the best thing one can do today is to assume no intercation at all and cut the packing fraction at some small value. There is also a more general argument that the relation derived in [4] must be wrong. The topological susceptibility is of $O(N_c^0)$, whereas the gluon condensate is of $O(N_c)$. (6) is consistent with $N_c \to \infty$ considerations only due to the presence of the $4/b$ factor.

5 The Mass of the η' meson

Let us now continue with the calculation of $m_{\eta'}$. The physical gluon condensate in the presence of light quarks is

$$\langle 0|N(0)|0\rangle_{phys} = (200\mathrm{MeV})^4 \quad . \tag{11}$$

Due to the presence of light quarks it is reduced by a factor $\alpha < 1$.

$$\langle 0|N(0)|0\rangle_{phys} = \alpha \langle 0|N(0)|0\rangle_{\Theta=0}^{no\ quarks} \tag{12}$$

Combining (1), (2), (5) and (12), we get the final formula for the η' mass is

$$m_{\eta'}^2 = \frac{4N_f}{f_\pi^2}\frac{12}{11N_c}\frac{1}{\alpha}\langle 0|N(0)|0\rangle_{phys} \tag{13}$$

One can see that $m_{\eta'}^2 \sim 1/N_c$ because f_π^2 and the gluon condensate are proportional to N_c. The largest uncertanty lies in the determination of α. Using again the instanton model, α is the determinant of the Dirac operator with the current quark masses replaced by effective masses:

$$\alpha = \prod_{i=u,d,s} 1.34 m_i^{eff}\rho \approx 0.4\ldots 0.7 \tag{14}$$

We have set the effective masses to the constituent masses

$$m_u^{eff} = m_d^{eff} = 300\ldots 350 \text{ MeV} \quad , \quad m_s^{eff} = 400\ldots 500 \text{ MeV} \tag{15}$$

and ρ to the value of the instanton liquid model ($\rho = 600 \text{ MeV}^{-1}$). This estimate is consistent with the estimate of [7]. Inserting $N_f = 3$ and $f_\pi = 132$ MeV into (13) we get

$$m_{\eta'} = 884 \pm 116\mathrm{MeV} \tag{16}$$

which is in good agreement to the experimental value of 958 MeV. This result in turn confirms the assumption, that the interaction between selfdual and anti-selfdual regions is small. The large uncertanty in α prevents more accurate statements, but (10) can definitely be excluded.

Using $m_{\eta'}$ as an input we can determine the gluon condensate in pure QCD

$$\frac{\alpha_s}{4\pi}\langle \mathrm{tr}_c GG\rangle^{no\ quarks} = (246 \text{ MeV})^4 \tag{17}$$

where we have again set N_f to 3.

6 Conclusions

We have calculated η' successfully in a model of seldual QCD. The factor $4/b$ in (6) is the essential term to get the correct N_c dependence of $m_{\eta'}$ and agreement with the experimental mass. The discussion has shown, that the η' channel can be an experimental device for testing the independence of selfdual and antiselfdual regions in QCD, which is an assumption in the simplest instanton models. It might turm out some day that the details of instanton models are wrong but the assumption of independent self(anti)dual regions remain valid.

References

[1] E.V. Shuryak, J.J.M. Verbaarschot: *Nucl.Phys. B410 (1993) 37 Nucl.Phys. B410 (1993) 55 Nucl.Phys. B412 (1994) 143*

[2] M. Hutter *Instantons in QCD and meson correlation functions LMU-95-01 München preprint, Submitted to Zeitschr.Phys.*

[3] E. Witten: *Nucl.Phys. B156 (1979) 269*

[4] D.I. Dyakonov, V.Yu. Petrov: *Nucl.Phys. B245 (1984) 259*; *B272 (1985B) 457*

[5] J.M.M. Verbaarschot: *Nucl.Phys. B362 (1991) 33*

[6] M.A. Shifman: *Phys.Rep. 209 (1991) 341*

[7] M.A. Shifman et al.: *"Instanton density modified by light quarks, gluon condensate"*; *Nucl.Phys. B163 (1980) 46, B191 (1981) 301*

[8] J.S. Bell, R. Jackiw; *Nuovo.Cim. A51 (1969) 47*
S.L. Adler, W.A. Bardeen: *Phys.Rev. 182 (1969) 1517*

[9] B.V. Geshkeinbein, B.L. Ioffe: *Nucl.Phys. B166 (1980) 340*

[10] G. 't Hooft: *Phys.Rev. D14 (1976) 3432*
C. Bernard: *Phys.Rev. D19 (1979) 3013*