INSTANTONS IN QCD

Theory and Application of the Instanton Liquid Model

Ph.D. Thesis of the Faculty for Physics
at the Ludwig Maximilians University Munich

Presented by
Marcus Hutter
Munich, 17^{th} of October, 1995

Inofficial translation from the German Ph.D. thesis

INSTANTONEN IN DER QCD
Theorie und Anwendungen des
Instanton-Flüssigkeit-Modells

1. Advisor: Prof. H. Fritzsch
2. Advisor: Prof. S. Theisen
Defense: 9 February 1996

Abstract

The current theory of strong interactions, the quantum chromodynamics (QCD), is a non-abelian gauge theory, based on the gauge group SU(3). Despite its formal similarity to QED there are significant differences. It was shown in 1973 already that the coupling constant g increases for large distances [61]. This gave hope for the possibility to explain quark and gluon confinement. Soon after, in 1975, non-trivial solutions of the Euclidian Yang Mills equations were found, nowadays called BPST instantons [15], which significantly influence the low energy structure of QCD. Many exact results are known for the 1-instanton vacuum [11, 18], whereby an interesting phenomenological result of it is the explicit breaking of the axial $U(1)$ symmetry [17]. On the other hand, a one instanton approximation, similar to a tree approximation in perturbation theory, cannot describe boundstates or spontaneous symmetry breaking. The next step was the analysis of exact [16] and approximate [14] multi-instanton solutions. There are two useful visualizations for these solutions. In one of these, instantons are interpreted as tunneling processes between different vacua. In the other interpretation, a solution describes an ensemble of extended (pseudo) particles in 4 dimensions.

In conventional perturbation theory one computes fluctuations around the trivial zero solution. The correct quantization process is to consider all classical solutions of the field equations and their fluctuations. In the path integral representation of QCD the partition function is, hence, dominated by an ensemble of extended particles (instantons) in 4 dimensions at temperature g^2. In the simplest case the partition function describes a diluted ideal gas of independent instantons. Unfortunately, this assumption leads to an infinite instanton density caused by large instantons, which obviously contradicts the assumption of a diluted gas. This problem is known as the infrared problem. The problem is avoided by assuming a repulsive interaction [24] which prevents the collapse. This is the model of a 4 dimensional liquid. Under certain circumstances the interaction can be replaced by an effective density. The Instanton Liquid Model in a narrow sense describes the QCD vacuum as a sum of independent instantons with radius $\rho = (600\text{MeV})^{-1}$ and effective density $n = (200\text{MeV})^4$. The correctness of this model is still being intensively investigated. So far the model is essentially justified by its phenomenological success.

Numerical simulations of the Instanton Liquid Model allowed to determine a number of hadronic quantities, especially meson masses, baryon masses, hadron wave functions, and condensates [27, 28].

For computing the quark propagators and the meson correlators there are also analytical methods. The most important predictions are probably the breaking of the chiral symmetry (SBCS) in the axial triplet channel [25] and the absence of Goldstone bosons in the axial singlet channel.

The largest part of this thesis is devoted to extending the analytical methods and to evaluating the results in (semi)analytical form.

The meson correlators (also called polarization functions) will be computed in the Instanton Liquid Model in zeromode and $1/N_c$ approximation, whereby dynamic quark loops will be taken into account. A spectral fit allows the computation of the masses of the σ, ρ,

ω, a_1 and f_1 mesons in the chiral limit. A separate consideration also allows computation of the η' mass. The results coincide on a 10% level with the experimental values. Furthermore, determining the axial form factors of the proton, which are related to the proton spin (problem), will be attempted. A gauge invariant gluon mass for small momentum will also be computed.

The thesis ends with several predictions which do not rely on the Instanton Liquid Model. In the 1-instanton vacuum a gauge invariant quark propagator will be computed and compared to the regular and singular propagator. Rules for the choice of a suitable gauge, especially between regular or singular, will be developed. A finite relation between the quark condensate and the QCD scale Λ will be derived, whereby neither an infrared cutoff, nor a specific instanton model will be used.

Contents

The chapters marked with * are more technical in nature.

List of Tables

Chapter 1

Introduction

The fundamental theories which describe the observed interactions are all gauge theories. They describe gravitation, electroweak and strong interaction. In the quantized version the forces between particles (fermions) are mediated by gauge bosons. QCD describes the strong interaction between quarks and gluons. It is a SU(3) gauge theory with the Lagrangian[1]

$$\mathcal{L} = \frac{1}{4g^2} G^a_{\mu\nu} G^{\mu\nu}_a + \sum_{i=1}^{N_f} \bar{\psi}_i (i\not{D} + im_i)\psi. \tag{1.1}$$

Taking into account only the light quarks u, d and sometimes s and setting their mass to zero, one arrives at a theory with only one parameter, the gauge coupling constant g, which actually is no parameter due to dimensional transmutation. Despite the fact that QCD looks so simple (e.g. compared to electroweak interaction) it is very hard to solve this theory due to nonperturbative effects.

1.1 Methods to Solve QCD

To clarify the role of instantons in QCD let me first give a list of the most important methods used to tackle QCD, starting with very general methods applicable to any quantum field theory (QFT) and ending with more specific approaches:

- *Axiomatic field theory:* Wightman/Oswalder&Schrader have stated a set of Minkowskian/Euclidian axioms for vacuum correlators a general QFT should respect (analyticity, regularity, Lorentz invariance, locality, ...). It is clear that the theorems derived from these axioms have to be very general because no Lagrangian is used [2].

[1] Throughout the entire work the Euclidian formulation of QCD is used [13].

- *Haag-Ruelle/LSZ Theory:* S-Matrix elements are related to vacuum correlators. The S-Matrix is the central object for particle phenomenology, containing such important information as cross sections, form factors, structure functions, ... Vacuum correlators are more suitable for theoretical studies because they avoid the need of explicitly constructing the Hilbert space, which is an extremely complicated task beyond perturbation theory.

- *Quantization:* One might think that quantization should not appear in a list of methods for solving QFT because it is the method to obtain and define a QFT. On the other hand, besides canonical quantization there are other ways of quantizing a theory. The most popular is the path integral quantization [8]. In textbooks for particle physics it is usually only used as an abbreviation to derive theorems more quickly. In general (beyond perturbation theory), different quantization methods lead to different physical and mathematical insights and different methods to solve the theory. Variants of the path integral quantization are the random walk quantization used in lattice theories, and stochastic quantization.

- *(Broken) Symmetry:* Every degree of freedom like spin, flavor and color is the possible origin of (approximate) symmetries like $SU(2N_f)$ or subgroups and $SU(3)$ gauge invariance. Conserved currents and Ward identities [4] can be obtained. In the case of light quarks, one further has approximate chiral symmetry leading to PCAC, axial Ward identities, current algebra theorems, soft pion physics,.... Furthermore, QCD with massless quarks possesses an anomalously broken scale invariance, which is the origin of the huge field of renormalization group techniques [5].

- *Perturbation theory:* Due to asymptotic freedom the coupling constant g decreases for high energies and perturbation theory in g is applicable. QCD can thus be solved for processes which involve only momenta of, say, more than 1 GeV. The small distance behaviour of vacuum correlators is, thus, calculable ($x \leq 0.2$ fm).

- *Operator Product Expansion (OPE):* An improvement of perturbation theory is to separate the small distance physics from large distance effects. The former is contained in the so-called Wilson coefficients calculated perturbatively. The latter nonperturbative effects are contained in a few vacuum or hadron expectation values of *local* operators which have to be determined from phenomenology or uncertain assumptions [13]. Vacuum correlators can be obtained up to distances of $x \leq 0.3 \ldots 0.5$ fm.

- *QCD Sum rules:* QCD sum rules are widely used to determine hadron masses and couplings. The general method is to assume the existence of certain hadrons and take a resonance+continuum ansatz in the Minkowskian region for some correlator. The Euclidian correlator is calculated from theory (OPE, lattice, instantons). Via dispersion relations one may match both in some Euclidian window by fitting the hadron parameters which leads to a prediction for them.

- *Effective Theories:* One may construct effective Lagrangians containing mesons and/or baryons more or less motivated by QCD or history or other physical branches.

Their parameters are determined from experiment or QCD as far as possible.

A variety of phenomena have been explained qualitatively and calculated quantitatively with the methods listed above, but the large distance behaviour of QCD is still unsolved. There are at least two problems belonging to this domain: chiral symmetry breaking and confinement. Up to now chiral symmetry breaking was assumed and the consequences such as Golstone bosons were discussed within this assumption. In OPE one takes the nonzero values of the quark and other condensates from experiment but has no possibility to predict them from theory. The quark condensate is the order parameter of chiral symmetry and a nonzero values indicates spontaneous breaking of this symmetry (SBCS). Confinement has also to be assumed.

Approaches to solve these problems are:

- *Lattice QCD:* In principle the method is very simple. The continuum is replaced by a fine lattice covering a large but finite volume. The path integral is thus replaced by a finite number of integrals evaluated numerically. All vacuum correlators can be obtained for arbitrary Euclidian distances. Confinement and SBCS have been shown and other hadronic parameters are obtained. In practice, lattice calculations are much less straightforward than this sketchy description might suggest [9].
- *Instantons:* As in lattice QCD one evaluates the Euclidian path integrals but now in semiclassical approximation. In addition to the global minimum of the QCD action $A_\mu^a = 0$ used by perturbation theory there are many other local minima called instantons which have to be taken into account. Inclusion of light quarks leads to an effective $2N_f$ quark vertex responsible for SBCS. Although confinement cannot be explained, a lot of hadron parameters can nevertheless be calculated [27, 28] suggesting that confinement is not essential for the properties of hadrons.

Often only a combination of the results from various approaches allows contact to phenomenology.

1.2 Contents

In this work I want to give a quantitative and systematic study of the implications of instantons to hadron properties. For an elementary introduction into the classical and semi-classical theory of solitons and instantons I recommend [11].

Chapter 2 is an introduction to the semiclassical evaluation of nontrivial integrals. After developing the method in the finite dimensional case the partition function of QCD is considered and the instanton liquid model is introduced.

In *Chapter 3* the propagator of a light quark is calculated. The approximations which have to be made are stated and discussed. It is shown that for one quark flavor the

$1/N_c$ expansion is exact. The constituent quark masses and the quark condensates are calculated for u,d and s quarks.

The same approximations are used in *Chapter 4* to calculate the 4 point functions. Special attention is payed to the singlet correlator where a chain of quark loops contributes and is *not* suppressed in the large-N_c limit. Within one and the same approximation we get Goldstone bosons in the pseudoscalar triplet correlator but no massless singlet boson.

In *Chapter 5* meson correlators are discussed and plotted. Employing a spectral ansatz it is possible to extract various meson masses and couplings. They are compared to the values obtained from extensive numerical studies of the instanton liquid [28] and to the experimental values.

In *Chapter 6* the axial anomaly is examined. The most interesting quantities in the axial singlet channel are the mass of the η' meson and the spin of the proton. A combination of the axial anomaly and the scale anomaly with ideas from instanton physics $m_{\eta'}$ can be determined with an accuracy of 10%. The focus is on the discussion of the proton spin and its computation in the Instanton Liquid Model.

In *Chapter 7* the ghost and gluon propagator in the Instanton Liquid Model will be computed. For small momentum the ghost mass is 340 MeV and the gluon mass is 480 MeV. The masses are independent of the gauge parameter ξ.

Chapter 8 attempts to predict several quantities without relying on Instanton Liquid Model. In regular gauge it is possible to derive a finite relation between the quark condensate and the QCD scale Λ. Since the results depends critically on the choice of the right gauge, the chapter starts with a detailed discussion on the choice of a gauge and with a calculation of a gauge independent propagator.

A summary of the results, new in this thesis, can be found in *Chapter 9*, followed by open questions, which deserve further investigation. The *Appendices* contain a list of used notation and explicit expressions for one instanton and its Dirac zeromode in singular, regular, and axial gauge. Furthermore, methods to compute Fourier transforms and convolutions of Lorentz covariant functions are described.

All *Figures* can be found at the end of this work before the *Reference* section (sorted with respect to topics).

The various chapters can be read independently of each other. Sections marked with * are more technical in nature.

Chapter 2

Theory of the Instanton Liquid

This chapter gives an introduction into the semiclassical evaluation of the QCD partition function. For this, the fluctuations around the classical solutions of the Euclidian equations of motion are separated into (approximately) Gaussian and collective coordinates (Section 2.2). In Section 2.3 the approximately Gaussian degrees of freedom will be integrated out. The classical solutions of the Yang Mills equations can be classified with respect to their topology $N \in \mathbb{Z}$ and are called multi-instanton solutions. For an introduction into the mathematically beautiful theory of instantons the reader should consult [16]. An elementary introduction into the classical and semiclassical theory of solitons and instantons can be found in [11]. The classical one-instanton solution and its corresponding semi-classical partition function will be presented in Section 2.4. The multi-instanton solutions can be reduced to the one-instanton case, in the case of well-separated instantons. Section 2.5 describes how quarks modify the partition function. In Section 2.6 the infrared problem will be discussed and the Instanton Liquid Model to "solve" this problem. This model is the basis of Chapters 3-7.

2.1 Separating Gaussian from non-Gaussian Degrees of Freedom *

In the path integral formulation solving QCD or any other QFT is equivalent to calculating the partition function Z including external source terms. Of course this is not a simple task because one has an infinite number of degrees of freedom. One way to tackle this problem is to reduce the number of degrees of freedom by integrating the (nearly) Gaussian degrees perturbatively leaving the non-Gaussian degrees, called collective coordinates, to be treated with other methods.

Let me first describe this method most generally without referring to QFT. Our aim is to calculate

$$Z = \int d^n x \, e^{-S[x]} \tag{2.1}$$

where S is some real valued function depending on the n dimensional vector $x \in I\!R^n$. Let the integral be dominated by values of x which lie in the vicinity of a k dimensional submanifold of $I\!R^n$ which may be parameterized by $x = f(\gamma)$, $\gamma \in I\!R^k$. Vectors in the tangent space of $f(\gamma)$ are called (approximate) zeromodes, because $S[x(\gamma)]$ is (nearly) constant in this direction. Usually $f(\gamma)$ represents some degenerate or approximate minima of S. In addition $f(\gamma)$ may contain points which do not contribute much to the functional integral, this will cause no error, but $f(\gamma)$ must not forget any significant points. In figure B.1 a two dimensional example is shown containing a river (one dimensional manifold) with steep mountains aside his banks. So the integral will be dominated by the shaded area. This area can be parameterized in the following way:

$$x = f(\gamma) + y \quad , \quad x, y \in I\!R^n \quad , \quad \gamma \in I\!R^k \quad . \tag{2.2}$$

To make this representation unique, we have to demand k linear extra conditions to y:

$$y \cdot g_i(\gamma) = c_i \quad , \quad 1 \le i \le k \tag{2.3}$$

E.g., $c_i = 0$ and $g_i(\gamma) = \partial f(\gamma)/\partial\gamma_i$ would fix y to be orthogonal to the "river" or equally stated: It disallows fluctuations in the non-Gaussian zeromode direction. Now every point x (at least in the vicinity of the river) can uniquely be described by y satisfying the condition (2.3) and by γ via (2.2). All we have to do now is to represent Z in terms of the fluctuation vector y and the collective coordinates γ. A convenient way is to introduce a Faddeev-Popov unit

$$1 = \int d^k\gamma \, d^n y \, \delta^k(y \cdot g_i(\gamma) - c_i) \, \delta^n(f(\gamma) + y - x) \, \Phi(x) \tag{2.4}$$

which serves as a definition of Φ. Inserting this into (2.1) and integrating over x yields

$$Z = \int d^k\gamma \, d^n y \, \delta^k(y \cdot g_i(\gamma) - c_i) \, \Phi(f(\gamma) + y) \, e^{-S[f(\gamma)+y]} \quad . \tag{2.5}$$

The y-integration in (2.4) can trivially be performed:

$$\Phi^{-1}(x) = \int d^k\gamma' \, \delta^k((x - f(\gamma'))g_i(\gamma') - c_i) \quad . \tag{2.6}$$

For $x = f(\gamma) + y$ the δ-function in the integral only contributes for $\gamma' = \gamma$. Thus we can expand the δ-argument up to linear order in $\gamma' - \gamma$

$$\begin{aligned} \Phi^{-1}(f(\gamma) + y) &= \int d^k\gamma' \delta^k \left(\sum_j (y \cdot \tfrac{\partial g_i(\gamma)}{\partial \gamma_j} - g_i(\gamma) \cdot \tfrac{\partial f(\gamma)}{\partial \gamma_j})(\gamma'_j - \gamma_j) \right) \\ &= |\det{}_{ij} \, (y \cdot \partial_j g_i - g_i \cdot \partial_j f)|^{-1} \quad {}^{1} \end{aligned} \tag{2.7}$$

$$\text{with} \quad y \cdot g_i(\gamma) = c_i \quad .$$

[1] In the following we will omit the absolute bars; when necessary they have to be reinstated.

Inserting Φ into (2.5) we get

$$Z = \int d^k\gamma\, d^n y\, \delta^k(y\cdot g_i(\gamma) - c_i)\, \det_{ij}\left(y\cdot\partial_j g_i(\gamma) - g_i(\gamma)\cdot\partial_j f(\gamma)\right)\, e^{-S[f(\gamma)+y]} \quad . \tag{2.8}$$

One may write Z in a slightly different form, usually used in QFT, because it is more suitable for semiclassical approximations. Z is independent of c_i and therefore, although the r.h.s. of (2.8) explicitly contains c_i, it is actually independent of it. We can smooth the δ-function by a further multiplication with

$$1 = (2\pi\xi)^{-k/2} \int d^k c\, e^{-\frac{1}{2\xi}\sum_i c_i^2} \quad . \tag{2.9}$$

The determinant can be written in the form

$$\det A = \int d\eta\, d\bar\eta\, e^{\bar\eta A \eta} \tag{2.10}$$

where η are anticommuting Grassmann variables (ghosts). Inserting (2.9) and (2.10) into (2.8) and performing the c-integration we finally get

$$\begin{aligned} Z &= (2\pi\xi)^{-k/2} \int d^k\gamma\, d^n y\, d\eta\, d\bar\eta\, e^{-S[f(\gamma)+y] - S_{gf}[y,\gamma] + S_{FPG}[y,\gamma,\eta,\bar\eta]} \quad , \\ S_{gf} &= \frac{1}{2\xi} y^T \left(\sum_i g_i(\gamma) g_i^T(\gamma)\right) y \quad , \\ S_{FPG} &= \sum_{ij} \bar\eta_i \left(y\cdot\partial_j g_i(\gamma) - g_i(\gamma)\cdot\partial_j f(\gamma)\right) \eta_j \quad . \end{aligned} \tag{2.11}$$

For a globally bijective transformation this representation of the partition function Z is still exact, for a locally bijective transformation the representation is exact in every order perturabtion theory, especially in semiclassical approximation. At this point the (approximately) Gaussian degrees of freedom y, η and $\bar\eta$ could be integrated out in semiclassical approximation. The partition function then reduces to an integral over the collective coordinates. In Section 2.3 this step will be performed directly for the QCD case.

2.2 Effective QCD Lagrangian in a Background Field *

Now it is time to return to QCD

$$Z = \int DA_\mu\, e^{-S_{YM}[A]} \quad , \quad S_{YM}[A] = \int dx\, \frac{1}{4g^2} G^a_{\mu\nu} G_a^{\mu\nu} \quad . \tag{2.12}$$

The background configurations which (approximately) minimize S_{YM} will be denoted by $\bar A_\mu(\gamma)$. A general gauge field can be written in the form

$$A_\mu = (\bar A_\mu(\gamma) + B_\mu)^\Omega \quad , \quad \gamma \in \mathbb{R}^k \tag{2.13}$$

where B_μ are fluctuations around the background and A_μ^Ω is a gauge transformed field

$$A_\mu^\Omega = SA_\mu S^\dagger + iS\partial_\mu S^\dagger \quad , \quad S = e^{i\Omega} \in SU(N_c) \quad . \tag{2.14}$$

As in the finite dimensional case we have to make this representation unique by introducing extra conditions,

$$D_\mu(\bar{A})B_\mu(x) = C(x) \quad , \tag{2.15}$$

to fix the gauge of the fluctuating field and

$$\int d^4x\, \psi_\mu^i(x;\gamma)B_\mu(x) = c_i \quad , \quad 1 \le i \le k \tag{2.16}$$

to avoid fluctuations in (approximately) zero mode direction. The derivation of an effective action similar to (2.11) can now be performed in full analogy to the previous case with only some notational complication. There is the following correspondence:

$$\begin{array}{lcccccc} \text{finite example} & : & i & x & y & \gamma_i & g_i \\ \text{QCD} & : & i,x & A & B & \gamma_i, \Omega(x) & \psi_\mu^i \end{array} \quad . \tag{2.17}$$

The Faddeev-Popov unit has the form

$$1 = \int d^k\gamma\, D\Omega\, DB_\mu\, \delta(D_\mu(\bar{A})B_\mu)\delta^k(\int \psi_\mu^i B_\mu d^4x)\delta((\bar{A}_\mu + B_\mu)^\Omega - A_\mu)\Phi[A_\mu] \quad . \tag{2.18}$$

The steps to get an expression for Φ are now:

Add primes to γ, Ω and B, insert $A = \bar{A} + B$, linearize the last δ-argument around $B'_\mu = B_\mu$, perform the functional B'_μ integration and linearize the remaining δ arguments around $\gamma' = \gamma$ and $\Omega' = 0$. Omitting the details of this calculation one gets [25]

$$\Phi^{-1}(\bar{A}(\gamma) + B) = \int d^k\gamma'\, D\Omega' \delta^k(X_i)\, \delta(Y) \quad , \tag{2.19}$$

$$X_i = \int d^4x \sum_j \left(\psi_\mu^i(\gamma)\frac{\partial \bar{A}_\mu}{\partial \gamma_j} - \frac{\partial \psi_\mu^i(\gamma)}{\partial \gamma_j} B_\mu \right)(\gamma'_j - \gamma_j) + \psi_\mu^i D_\mu(\bar{A} + B)\Omega' \quad , \tag{2.20}$$

$$Y = \sum_j D_\mu(\bar{A} + B)\frac{\partial \bar{A}_\mu}{\partial \gamma_j}(\gamma'_j - \gamma_j) + D_\mu(\bar{A})D_\mu(\bar{A} + B)\Omega' \quad . \tag{2.21}$$

From (2.15), (2.16), (2.20) and (2.21) one can read off the form of the partition function Z:

$$Z = N(\xi) \int d^k\gamma\, DB_\mu\, D\eta\, D\bar{\eta}\delta^k(\int \psi_\mu^i B_\mu d^4x)e^{-S_{QCD}[\bar{A},B,\eta,\bar{\eta}]} \quad , \tag{2.22}$$

$$\begin{aligned} S_{QCD} &= S_{YM}[\bar{A}+B] - S_{gf}[\bar{A},B] + S_{FPG}[\bar{A},B,\eta,\bar{\eta}] \quad , \\ S_{YM} &= \int d^4x \frac{1}{4g^2} G^a_{\mu\nu} G^{\mu\nu}_a(\bar{A}+B) \quad , \\ S_{gf} &= \frac{1}{2\xi} \int d^4x \, (D_\mu(\bar{A})B_\mu)^2 \quad , \\ S_{FPG} &= \sum_{ij} \bar{\eta}_i \left[\int d^4x \, \psi^i_\mu(\gamma) \frac{\partial \bar{A}_\mu}{\partial \gamma_j} - \frac{\partial \psi^i_\mu(\gamma)}{\partial \gamma_j} B_\mu \right] \eta_j \\ &+ \sum_i \int d^4x \, \bar{\eta}_i \psi^i_\mu D_\mu(\bar{A}+B)\eta(x) + \int d^4x \sum_j \bar{\eta}(x) D_\mu(\bar{A}+B) \frac{\partial \bar{A}_\mu}{\partial \gamma_j} \\ &+ \int d^4x \, \bar{\eta}(x) D_\mu(\bar{A}) D_\mu(\bar{A}+B)\eta(x) \quad . \end{aligned} \tag{2.23}$$

S_{QCD} does not depend on the gauge parameter Ω. For this reason the Ω integration can be absorbed in the normalization factor $N(\xi)$. $\eta(x)$ are the usual ghost fields originating from the gauge fixing. For every extra condition (2.16) one gets an additional ghost variable η_i. For $\bar{A}=0$ and no extra condition ($k=0$) the action given above just reduces to the usual QCD action including Faddeev-Popov ghosts in R_ξ gauge

$$S_{gf} = \frac{1}{2\xi} \int d^4x \, (\partial_\mu B_\mu)^2 \quad , \quad S_{FPG} = \int d^4x \, \bar{\eta}(x) \partial_\mu D_\mu(B) \eta(x) \quad . \tag{2.24}$$

Note that the action (2.22) is still exact with the non-harmonic degrees of freedom γ_i now separated from the hopefully more Gaussian ones, B_μ and η.

For small coupling g it is now possible to establish Feynman rules from (2.23) in analogy to the case with no background. For this one has to know the "free" gluon, ghost and quark propagator in a given background $\bar{A}$. If $\bar{A}$ is a non constant field even this is a very complicated task in contrast to usual perturbation theory around $\bar{A}=0$. For a multi-instanton configuration explicit expressions for the gluon and ghost propagator are derived in [18].

2.3 The Semiclassical Limit *

Before developing perturbation theory to all orders it is wise to study the semiclassical limit where one keeps only terms up to quadratic order in the fields. In QCD (and many other field theories) this is equivalent to lowest order perturbation theory, but around a very nontrivial background!

Up to now we have not specified ψ^i_μ. A natural choice would be $\psi^i_\mu = \partial \bar{A}_\mu / \partial \gamma_i$ to fix the fluctuations to be orthogonal to the zero modes. Somewhat more convenient is to bring ψ^i_μ in background gauge:

$$\psi^i_\mu = \left(\frac{\partial \bar{A}_\mu}{\partial \gamma_j} \right)^\Omega \quad \text{with } \Omega \text{ such that} \quad D_\mu(\bar{A}) \psi^i_\mu = 0 \quad . \tag{2.25}$$

Furthermore we assume that $\bar{A}$ minimizes the gauge action S_{YM} which is true for widely separated instantons thus neglecting linear terms S_{QCD}.

Up to quadratic order in the fields one has

$$\begin{aligned} S_{QCD} &= S_{YM}[\bar{A}] + \int d^4x \, \frac{1}{2g^2} B_\mu K_{\mu\nu}(\bar{A}) B_\nu + \int d^4x \, \bar{\eta}(x) D^2(\bar{A}) \eta(x) \\ &+ \sum_{ij} \bar{\eta}_i \psi^i_\mu \frac{\partial \bar{A}_\mu}{\partial \gamma_j} \eta_j + \int d^4x \sum_j \bar{\eta}(x) D_\mu(\bar{A}) \frac{\partial \bar{A}_\mu}{\partial \gamma_j} \eta_j + O(\text{field}^3) \quad , \end{aligned} \tag{2.26}$$

$$K_{\mu\nu} = -D^2 \delta_{\mu\nu} + 2iG_{\mu\nu} + (1 - \frac{1}{\xi}) D_\mu D_\nu \quad , \quad G_{\mu\nu} = F^c G^c_{\mu\nu} \quad , \quad (F^c)_{ab} = i f_{acb} \quad . \tag{2.27}$$

Performing the integration over gauge fields and ghosts one gets an effective action depending only on the collective coordinates γ_i:

$$Z = \int d^k\gamma \, e^{-S_{eff}[\gamma]} \tag{2.28}$$

$$e^{-S_{eff}[\gamma]} = \det_{ij} \left(\psi^i_\mu(\gamma) \frac{\partial \bar{A}_\mu}{\partial \gamma_j} \right) \frac{\text{Det}(-D^2(\bar{A}))}{(\text{Det}' K'_{\mu\nu}(\bar{A}))^{1/2}} e^{-S_{YM}[\bar{A}]} \tag{2.29}$$

The δ-function in (2.22) causes a restriction of the gauge field fluctuation to be orthogonal to ψ^i_μ. $K'_{\mu\nu}$ is defined as $K_{\mu\nu}$ projected to the space orthogonal to ψ^i_μ, Det' takes into account all eigenvalues of $K'_{\mu\nu}$ except the k zeromodes caused by the projection.

2.4 Instantons in QCD

The solutions of the classical Yang-Mills equations of motion can be classified w.r.t. their topology $N \in \mathbb{Z}$ and are called N instantons[2] solutions [16]. The 1-instanton solution has the well known form

$$\begin{aligned} A^a_{I\mu}(x) &= O^{ab}_I \eta^{Q_I}_{b\mu\nu} \frac{(x - z_I)_\nu}{(x - z_I)^2} \frac{2\rho^2}{(x - z_I)^2 + \rho^2} && \text{in singular gauge} \\ A^a_{I\mu}(x) &= O^{ab}_I \eta^{-Q_I}_{b\mu\nu} \frac{2(x - z_I)_\nu}{(x - z_I)^2 + \rho^2} && \text{in regular gauge} \\ \gamma_I = (z_I, O_I, \rho_I, Q_I) &= (\text{location, orientation, radius, topological charge}) \end{aligned}$$

The instanton parameters γ_I reflect the symmetries of the Lagrangian (translation, rotation, scale invariance, parity). It was a great achievement of 't Hooft to compute the functional determinant in the 1-instanton background. To get a finite results Z_I has to be

[2] To simplify notations we will treat instantons and anti-instantons on the same footing. Both will be called instantons and are distinguished by their topological charge $Q_I = \pm$ if necessary.

normalized to the $\bar{A}^a_\mu = 0$ case, regularized and renormalized. The final result of the very complicated calculation [17] for the partition function in semiclassical approximation is

$$\begin{aligned}
\left(\frac{Z_I}{Z_0}\right)_{reg} &= \int d\gamma_I\, D(\rho_I) \;=\; \frac{1}{2}\sum_{Q_I=\pm 1}\int d^4 z_I dO_I d\rho_I\, D(\rho_I) \;=\; V_4\int_0^\infty d\rho\, D(\rho) \;=\; V_4 \bar{D} \\
D(\rho) &= \frac{C_{N_c}}{\rho^5} S_0^{2N_c} e^{-S_1(\rho)} \;=\; C_{N_c}\rho^{-5} S_0^{2N_c}(\rho\Lambda)^b \\
C_{N_c} &= \frac{4.6 e^{-1.679 N_c}}{\pi^2 (N_c-1)!(N_c-2)!} \\
S_0 &= \frac{8\pi^2}{g_0^2} \quad , \quad S_1(\rho) \;=\; \frac{8\pi^2}{g_1^2(\rho)} \;=\; b\ln\frac{1}{\rho\Lambda} \quad , \quad \Lambda = \Lambda_{PV} \\
S_2(\rho) &= \frac{8\pi^2}{g_2^2(\rho)} \;=\; b\ln\frac{1}{\rho\Lambda} + \frac{b'}{b}\ln\ln\frac{1}{\rho\Lambda} + O\left(\frac{1}{\ln\frac{1}{\rho\Lambda}}\right) \quad ,
\end{aligned}$$

$D(\rho)$ is the density of instantons of radius ρ, $g(\rho)$ the running coupling constant in 1/2-loop approximation, $b = \frac{11}{3}N_c$ and $b' = \frac{17}{3}N_c^2$. S_0 is the classical action of an instanton, and g_0 is the bare (unrenormalized) tree-level coupling constant. One can obtain an estimate for g_0 by replacing g_0 with the running coupling constant at a suitable scale. This unlucky situation can be avoided by using the 2-loop expression for $D(\rho)$, i.e. by replacing S_0 with S_1 and S_1 with S_2. But this replacement is an improvement only for small coupling. When ρ reaches the QCD scale Λ one should confine oneself to the tree-level or 1-loop approximation in order to obtain useful results, since the perturbation series is only asymptotically convergent.

2.5 Quarks

Additional fields coupled in a gauge invariant way to the gluon field can simply be incorporated by adding the appropriate Lagrangian with gauge field A replaced by $\bar{A}+B$ and performing the functional integration over the new fields. So every quark contributes an extra factor

$$\int D\Psi D\bar{\Psi} e^{-\int dx\, \bar{\Psi}(i\not{D}+im)\Psi} = \mathrm{Det}(i\not{D}+im) \tag{2.30}$$

to the partition function Z where

$$iD_\mu = i\partial_\mu + \bar{A}_\mu + B_\mu \tag{2.31}$$

is the covariant derivative. In the semiclassical approximation B_μ can be set to zero. $D(\rho)$ has to be multiplied by the fermionic factor

$$F(m\rho) = \begin{cases} 1.34 m\rho(1+m^2\rho^2\ln(m\rho)+\ldots) & \text{for} \quad m\rho \ll 1 \\ 1 - \frac{2}{75 m^2\rho^2} + \ldots & \text{for} \quad m\rho \gg 1 \end{cases} \tag{2.32}$$

and b and b' are now

$$b = \frac{11}{3}N_c - \frac{2}{3}N_f \quad , \quad b' = \frac{17}{3}N_c^2 - \frac{13}{3}N_c N_f + \frac{1}{2}\frac{N_f}{N_c} \quad . \tag{2.33}$$

2.6 The Instanton Liquid Model

The sum of well-separated instantons ($Q_I = \pm 1$) is also an approximate solution of the YM equations:

$$A = \sum_{I=1}^{N} A_I \quad , \quad S[A] \approx NS_0$$

The partition function of this so called instanton gas is

$$Z = \sum_{N=0}^{\infty} Z_N \quad , \quad Z_N \approx \frac{1}{N!}(V_4 \bar{D})^N$$

The sum is dominated by instanton configuration with density $N/V = \bar{D}$. Unfortunately $\bar{D}$ is infinite and the assumption of a diluted gas turns out to be wrong. The probability of small size instantons is low because $D(\rho)$ vanishes rapidly for small distances. On the other hand for large distances $D(\rho)$ blows up and soon gets large. This is the origin of the infrared problem which made a lot of people no longer believing in instanton physics. Those who were not deterred by that have thought of the following outcome [27]. For larger and larger distances, the vacuum gets more and more filled with instantons of increasing size. At some scale the instanton gas approximation breaks down and one has to consider the interaction between instantons which might be repulsive to stabilize the medium. The stabilization might occur at distances at which a semiclassical treatment is still possible and at densities at which the various instantons are still well separated objects - say - not much deformed through their interaction. So there is a narrow region of allowed values for the instanton radius. This picture of the vacuum is called the instanton liquid model. The idea has been confirmed in the course of years by very different approaches:

- Infrarot-Cutoff [14]
- Hardcore assumption [24]
- Variational Approach [25]
- Numerical studies [27]
- Phenomenological success [28]

The simplest suggestion is to introduces a cutoff ρ_c and to ignore large instantons [14]:

$$\bar{D}_{\rho_c} = \int_0^{\rho_c} d\rho \, D(\rho)$$

The cutoff has to be chosen sufficiently small such that the space time fraction f filled with instantons is smaller than 1 in order to justify the model of a diluted gas.

$$f = \frac{2}{N_c} \int_0^{\rho_c} d\rho \, \frac{1}{2} \pi^2 \rho^4 D(\rho) \quad < \quad 1$$

This simple cutoff procedure can be improved by introducing a scale invariant (hardcore) repulsion between instantons, which effectively suppresses large instantons [24]. This procedure has the advantage of respecting the scaling Ward identities which are otherwise violated by the simple cutoff ansatz In [25] such an repulsion has been found leading to a phenomenologically welcomed packing fraction. Unfortunately this repulsion is an artifact of the sum-ansatz as has been shown by [20]. Therefore the infrared problem is still unsolved.

Nevertheless it is possible to make successful prediction by simply assuming a certain instanton density and some average radius. It seems that the vacuum can be described by effectively independent instantons of size $\rho = 600\,\mathrm{MeV}^{-1}$ and mean distance $L_0 = 200\,\mathrm{MeV}$. The integral instanton density is fixed by the experimentally known gluon condensate [57]:

$$n = N/V_4 = 1/L_0^4 = \frac{1}{32\pi^2} < G^a_{\mu\nu} G^{\mu\nu}_a >= (200\,\mathrm{MeV})^4_{exp.} \tag{2.34}$$

In Chapter 3 it is shown that this relation is not modified by light quarks. The ratio L_0/ρ is estimated in different works to be

$$(L_0/\rho)_{theor.} = 3.0 \ldots 3.2 \tag{2.35}$$

The Instanton Liquid Model is therefore defined by the following assumption on $D(\rho_I)$:

$$D(\rho_I) = n\delta(\rho_I - \rho) \quad , \quad n = (200 \text{ MeV})^4 \quad , \quad \rho = 600 \text{ MeV}^{-1} \quad . \tag{2.36}$$

The model describes very successfully the physics of light hadrons. Extensive numerical studies can be found in [27, 28].

In high energy processes with momentum transfer p of $1 - 10$ GeV, $D(\rho)$ is often multiplied with function, which is sharply peaked around $\rho \sim p^{-1}$. The integral over ρ is then dominated by small instantons and is infrared convergent. The results are, hence, independent of the infrared cutoff. No additional assumptions have to be made. A similar phenomenon allows in Chapter 8 to derive a relation between the quark condensate and the QCD scale Λ without model assumptions. All other chapters are based on the Instanton Liquid Model.

Chapter 3

Light Quark Propagator

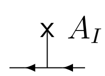
In this chapter the average quark propagator in the multiinstanton background will be calculated. The methods developed in [25] will be extended by taking into account dynamical quark loops. In the first section the instanton background is treated as a classical external perturbation, but the background field is not small (e.g. in the coupling g) and therefore we have to sum up *all* Feynman graphs. This is possible in the case of one quark flavor within the so-called zeromode approximation. This is possible due to an exact cancellation of certain graphs and by renormalizing the instanton density. As a byproduct we answer positively the question whether it is allowed to identify the gluon condensate with the instanton density in the presence of dynamical quarks. The quark condensate and a constituent quark mass are extracted from the quark propagator in section 3.8. In the last section it is shown that the case of two or more quark flavors can be reduced to the one flavor case in the limit of $N_c \to \infty$. The results obtained in the one flavor case are therefore still valid when making this further approximation.

3.1 Perturbation Theory in the Multi-instanton Background *

It is well known how to calculate correlators in the presence of an external classical gauge field at least as perturbation series in powers of the external field $A^a_\mu(x)$. In the case of QCD (or more accurately in classical chromodynamics) within the instanton liquid model the external field is a sum of well separated scatterers $A = \sum_I A_I$ called instantons with a fixed radius ρ and distributed randomly and independently in Euclidian space.

For a while we will restrict ourselves to the case of one quark flavor and ignore gluon loops. The Euclidian Feynman rules have the following form:

$$= \frac{1}{\not p + im} = S_0 \quad ,$$



$$= \not A_I \quad . \tag{3.1}$$

In operator notation,

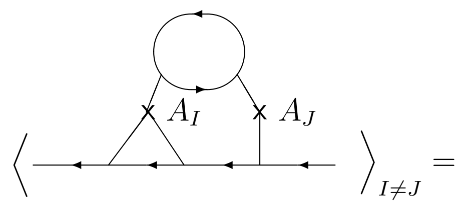
$$\langle p|S_0|q\rangle = \frac{1}{\not{p}+im}\bar{\delta}(p-q) \quad , \quad \langle x|\not{A}_I|y\rangle = \not{A}_I(x)\delta(x-y) \quad , \tag{3.2}$$

$$\bar{\delta}^d(\cdots) := (2\pi)^d\delta(\cdots) \quad , \quad \int \bar{d}^d p := \int \frac{d^d p}{(2\pi)^d} \quad ,$$

graphs are simply alternating chains of S_0 and A_I. To average a graph over the instanton parameters one has to perform the following integration for each instanton:

$$\langle \ldots \rangle_I = \frac{N}{V_4}\int d\gamma_I \ldots = \frac{N}{2}\sum_{Q_I=\pm}\frac{1}{V_4}\int d^4 z_I \int dO_I \ldots \quad . \tag{3.3}$$

For example



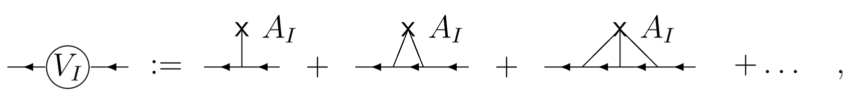
$$= -\frac{N^2}{V_4^2}\int d\gamma_I d\gamma_J \langle p|S_0\not{A}_I S_0 \not{A}_I S_0 \not{A}_J S_0|q\rangle \mathrm{Tr}(S_0\not{A}_J S_0 \not{A}_I) \quad . \tag{3.4}$$

The origin of the factor N^2 is the summation over all pairs of different instantons (I, J) yielding a factor $N(N-1) \approx N^2$. The quark loop is the origin of the minus sign and of the functional trace "Tr".

3.2 Exact Scattering Amplitude in the one Instanton Background *

Perturbation theory is suitable to study scattering processes. To achieve chiral symmetry breaking or bound states one has to sum up infinite series of a subclass of graphs or solve Schwinger Dyson or Bethe Selpeter equations.

The first thing we can do is to sum up successive scatterings at one instanton



$$V_I := \not{A}_I + \not{A}_I S_0 \not{A}_I + \not{A}_I S_0 \not{A}_I S_0 \not{A}_I + \ldots = S_0^{-1}(S_I - S_0)S_0^{-1} \tag{3.5}$$

where

$$S_I = (S_0^{-1} - \not{A}_I)^{-1} \tag{3.6}$$

is the quark propagator in the one instanton background.

3.3 Zero Mode Approximation

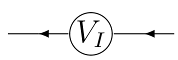
The quark propagator in the one instanton background can be represented in terms of the eigenvalues λ_i and eigenfunctions ψ_i of the Dirac operator

$$(i\partial\!\!\!/ - A\!\!\!/_I)\psi_i = \lambda_i \psi_i \quad \Longrightarrow \quad \langle x|S_I|y\rangle = \sum_i \frac{\psi_i(x)\psi_i^\dagger(y)}{\lambda_i + im} \quad . \tag{3.7}$$

There is one zero eigenvalue $iD\!\!\!\!/\,\psi_I = 0$ (↑Appendix A.2) which makes the propagator singular in the chiral limit:

$$S_I = \frac{|\psi_I\rangle\langle\psi_I|}{im} + S_I^{NZM} \quad . \tag{3.8}$$

In the so called zero mode approximation one replaces the non zero mode part by the free propagator



$$= S_I - S_0 \approx \frac{|\psi_I\rangle\langle\psi_I|}{im} \quad . \tag{3.9}$$

Although this approximation is good for large as well as for small momenta it may be bad for intermediate ones, but what is more important is the fact that it is a wild approximation and so might violate general theorems like Ward identities. In contrast, all other approximations we make are of systematic nature respecting all known symmetries of QCD.

- semiclassical approximation (systematic)
- multi-instanton background ("systematic")
- large N_c expansion (systematic) (see next section)
- zero mode approximation (wild)

Note that every choice of a background gauge field is "systematic" in the sense of respecting the symmetries of QCD as long as the background satisfies these symmetries on average.

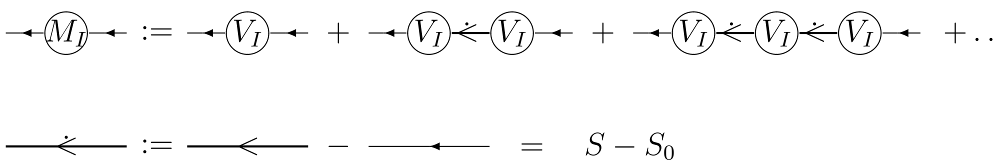
In the following sections we will see that the advantage of the zeromode approximation is so great that we cannot disregard this simplification.

3.4 Effective Vertex in the Multi-instanton Background *

In Section 3.2 the exact scattering amplitude V_I of the 1-instanton vacuum has been computed. In this Section a formal expression for the exact scattering amplitude M_I at one instanton in a multi-instanton bath will be derived.

Let us consider a quark line with two scatterings at V_I and insert in between a number of instantons which differ from I and from all other instantons occurring elsewhere in the graph. This enables us to average over these enclosed instantons independently from the rest of the graph. Summation over all possible insertions with at least one instanton just yields the exact quark propagator minus the free propagator. Remember that direct repeated scattering at A_I is already included in V_I.

Let us define



(3.10)

$$\begin{aligned} M_I &= V_I + V_I(S-S_0)V_I + V_I(S-S_0)V_I(S-S_0)V_I + \dots \\ &= V_I + V_I(S-S_0)M_I \end{aligned}$$

This equation can be solved for M_I with the following ansatz:

$$M_I = \frac{1}{i\mu} S_0^{-1} |\psi_I\rangle\langle\psi_I| S_0^{-1} \tag{3.11}$$

Inserting M_I and V_I into (3.10) we get

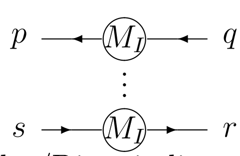
$$\frac{1}{i\mu} S_0^{-1}|\psi_I\rangle\langle\psi_I|S_0^{-1} = \frac{1}{im}(1 + \frac{1}{i\mu}\langle\psi_I|S_0^{-1}(S-S_0)S_0^{-1}|\psi_I\rangle)S_0^{-1}|\psi_I\rangle\langle\psi_I|S_0^{-1} \tag{3.12}$$

$$\Longrightarrow \quad \mu = m + i\langle\psi_I|S_0^{-1}(S-S_0)S_0^{-1}|\psi_I\rangle \quad . \tag{3.13}$$

The 1-instanton bath causes a replacement of the current mass m with an effective mass μ.

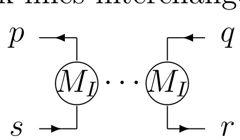
3.5 A Nice Cancellation *

It is possible to arrange the graphs in such a way that every M_I occurs only once. Consider a graph containing two scattering processes M_I at the same instanton. The interesting part of the graph has the following form



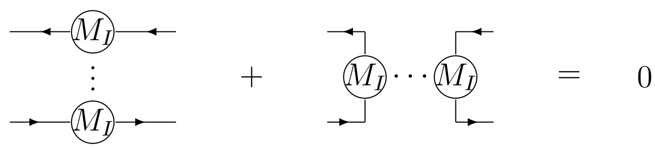
$$= {}^{\alpha_p}_{i_p}\Big[\frac{\psi_I(p)\psi_I^\dagger(q)}{i\mu}\Big]^{\alpha_q}_{i_q}\,{}^{\alpha_r}_{i_r}\Big[\frac{\psi_I(r)\psi_I^\dagger(s)}{i\mu}\Big]^{\alpha_s}_{i_s} \tag{3.14}$$

α_p/i_p are the color/Dirac indices at the quark leg with momentum p. In a physical process in addition to the graph containing the above subgraph there exists another graph with only two quark lines interchanged.



$$= -\,{}^{\alpha_p}_{i_p}\Big[\frac{\psi_I(p)\psi_I^\dagger(s)}{i\mu}\Big]^{\alpha_s}_{i_s}\,{}^{\alpha_r}_{i_r}\Big[\frac{\psi_I(r)\psi_I^\dagger(q)}{i\mu}\Big]^{\alpha_q}_{i_q} \tag{3.15}$$

As usual, the interchange of two quark lines causes a minus sign in the amplitude. Inspecting the two expressions, we see that they coincide except for the sign, thus there exists a complete cancellation



(3.16)

Whenever an M_I occurs twice or more than twice in a graph there exists another graph with opposite sign. Both contributions cancel each other and can be ignored. So a quark can scatter only once at every instanton. This can be seen in another way: Because of Fermi statistics every state can be occupied only once, and there is only one state for each quark in the zero mode approximation, namely the zeromode.

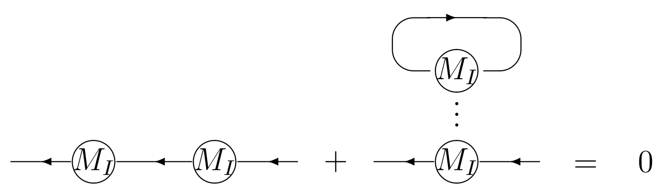
There are two equivalent descriptions of Feynman graphs:

1. Draw all topologically distinct graphs with non-numbered vertices and assign a symmetry factor to each graph,
2. Draw all topologically distinct graphs with numbered vertices and assign a factor $1/V$, where V is the number of vertices.

If all possible graphs containing M_I are allowed it is not difficult to see within the second description that they can really be paired as stated above.

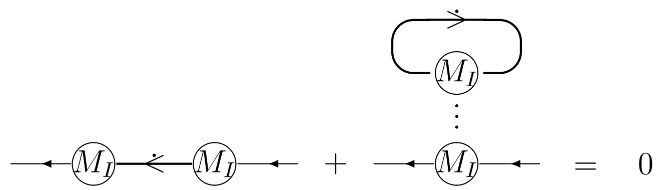
3.6 Renormalization of the Instanton-Density *

Up to now the cancellation is incomplete because not all graphs are allowed. Consider e.g.



not allowed *not allowed*

As in the case of V_I both graphs are not allowed. Another example is



not allowed *most general tadpole*

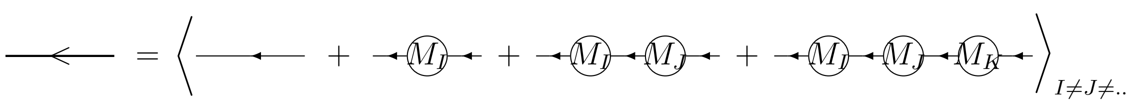
It is a general fact that all disallowed graphs can be paired with other disallowed graphs or with tadpoles and vice versa.

What we have to do is to "disallow" all tadpole graphs. Every M_I can be surrounded by tadpoles which contribute with a universal multiplicative factor which can be absorbed in a redefinition of the instanton density n_R. Using this renormalized density n_R the pairing is now perfect and the statement "every M_I occurs only once" becomes true.

One further can show that in the presence of dynamical quarks this renormalized density has to be identified with the gluon condensate instead of the "bare" density because the same tadpoles contribute to the gluon condensate too. The often asked question, whether in the presence of light quarks the gluon condensate can be identified with the instanton density, can be answered, in the above sense, with yes!

3.7 Selfconsistency Equation for the Quark Propagator *

Quark loops are no longer possible because they cannot be connected to another part of the graph via a common instanton. All graphs which can contribute to the propagator are chains of different $M_I's$.



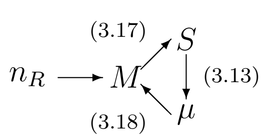
The $M_I's$ can be averaged independently

$$M(p) := i\langle M_I \rangle = \frac{n_R}{2\mu} p^2 \varphi'^2(p) \quad , \tag{3.17}$$

where φ' is defined in appendix A. The resulting expression for the propagator now has the form

$$S = S_0 + S_0 \frac{M}{i} S_0 + S_0 \frac{M}{i} S_0 \frac{M}{i} S_0 + \ldots = (S_0^{-1} + M)^{-1} \tag{3.18}$$

where $M = M(p)$ is the momentum dependent mass defined above. There is just one thing to do: We have to solve the circular dependence



but μ is just a number, which makes the solution very simple. Inserting (3.17) and (3.18) into (3.13) one gets the following equation for μ

$$\mu = m + \int d\!\!\!^{-}{}^4 p \, \frac{2\varphi'^2(p) M_p}{p^2 + (m + M_p)^2} (p^2 + m(m + M_p)) \tag{3.19}$$

which may be solved numerically for different current masses m.

3.8 Some Phenomenological Results

In the chiral limit (3.19) reduces to

$$\mu^2 = n_R \int đ^4p \, \frac{p^4 \varphi'^4(p)}{p^2 + M_p^2} = \alpha n_R \rho^2 + O(n_R^2) \tag{3.20}$$

μ^2 is proportional to n_R and thus M is proportional to $\sqrt{n_R}$ in contrast to a linear dependence on n_R obtained from a naive density expansion.

In the last expression the denominator has been expanded in the density and

$$\alpha = \rho^{-2} \int đ^4p \, p^2 \varphi'^4(p) = 6.6 \tag{3.21}$$

is a universal number. For the standard values of n_R and ρ one gets

$$\begin{aligned} \mu_0^2 &= 6.6 n_R \rho^2 = (100\text{MeV})^2 \quad , \\ M(p=0) &= 7.7 \rho \sqrt{n_R} = 300\text{MeV} \quad . \end{aligned} \tag{3.22}$$

The exact solution of (3.20) which has been obtained numerically by iteration, differs from the leading density value by 15%:

$$M(0) = 345\text{MeV} \quad . \tag{3.23}$$

The momentum dependence of the quark mass is shown in figure B.2. The mass $m + M(p)$ may be interpreted as the mass of a constituent quark. At high energies it tends to the current mass, at low momentum chiral symmetry breaking occurs and the quark gets its constituent mass M(0). Note that this is not a pole mass but a virtual mass at zero momentum squared.

Let us now take into account a small current mass m formally of the order $\sqrt{n_R}$. The selfconsistency equation now reads

$$1 = \frac{m}{\mu} + \frac{\mu_0^2}{\mu^2} + O(n_R) \quad . \tag{3.24}$$

Solving it for μ leads to

$$\mu_0 \quad \le \quad \mu = \frac{1}{2} m + \sqrt{\frac{1}{4} m^2 + \mu_0^2} \quad \le \quad m + \mu_0 \tag{3.25}$$

For the strange quark μ is increased by a factor of 2:

$$\mu(m_s = 150\text{MeV}) = 200\text{MeV} \tag{3.26}$$

It is interesting that $m_s + M(0)$ remains to be 300MeV. For zero momentum the increase of the current mass is just compensated by an equal decrease of the dynamical mass $M(0)$.

From the propagator one can obtain the quark condensate

$$\langle\bar{\psi}\psi\rangle := \lim_{x\to 0} \mathrm{tr}_{CD}(S(x) - S_0(x)) = N_c \int \bar{d}^4p\, \mathrm{tr}_D(S(p) - S_0(p)) \quad . \tag{3.27}$$

In leading order in the density one gets

$$i\langle\bar{\psi}\psi\rangle = \frac{n_R N_c}{\mu} = \langle G^a_{\mu\nu} G^{\mu\nu}_a\rangle / 32\pi^2\mu \quad . \tag{3.28}$$

This leads to the following condensates for u,d and s quarks:

$$i\langle\bar{u}u\rangle = i\langle\bar{d}d\rangle = (250\mathrm{MeV})^3 \quad , \quad \langle\bar{s}s\rangle = 0.5\langle\bar{u}u\rangle \quad . \tag{3.29}$$

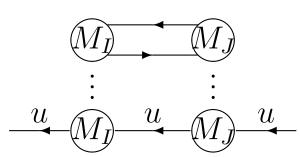
For heavy quarks there exists a similar relation

$$i\langle\bar{\psi}\psi\rangle = \langle G^a_{\mu\nu} G^{\mu\nu}_a\rangle / 48\pi^2 m + O(m^{-3}) \quad , \tag{3.30}$$

which leads within 10% to the same value for the strange quark condensate. This nicely confirms the hypothesis that the strange quark can be treated as a light quark as well as a heavy quark. This hypothesis is used in heavy to light quark matching formulas.

3.9 Large N_c expansion *

Consider now the case of N_f light quark flavors $u, d, s, \ldots$. The discussion of the one flavor case in the previous sections can be copied up to the pairing and cancelation of graphs which contain more than one M_I (3.16). This is still true in the multiple flavor case but now both quark lines in (3.16) must have the same flavor because M_I always connects quarks of the same flavor. So we have the theorem: "every M_I occurs only once for each flavor". From this point on the discussion of the one flavor case breaks down because there are now graphs contributing to the propagator containing quark loops. The simplest new contribution has the form



Is this contribution small in some sense ? Yes it is ! Quark loops are suppressed by a factor $1/N_c$. In perturbative context this is extensively discussed in [62], in instanton physics it was first used by [25]. Although 1/3 is not a very small number the large N_c expansion seems to be a good approximation in various cases.

Consider a graph and add to it a new quark loop consisting of

N	new instantons	$S \geq N$.
S	instanton scatterings	

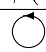
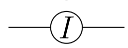
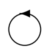
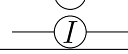
Parameters	$n_R, \rho, N_c, (g_R)$
Instanton density	$n = N/V_4 = n_R N_c \approx (200\text{MeV})^4$
Instanton radius	$\rho \approx (600\text{MeV})^{-1}$
Number of colors	$N_c = 3$
Coupling constant	$g = g_R/N_c$ (not used in semicl. limit)
Gluon condensate	$\langle GG \rangle = \langle G^a_{\mu\nu} G^{\mu\nu}_a \rangle / 32\pi^2 =: n_R N_c$
Quark condensate	$\langle \bar\psi\psi \rangle \approx 0.39\, N_c \rho^{-1} \sqrt{n_R} \approx (253\text{MeV})^3$
Constituent mass	$M(p) \sim \rho\sqrt{n_R}$
Quark mass	$M_{quark}(0) \approx 7.7\, \rho n_R^{1/2} \approx 300\text{MeV}$
Gluon mass	$M_{gluon}(0) \approx 10.9\, \rho n_R^{1/2} \approx 420\text{MeV}$[40]
Ghost mass	$M_{ghost}(0) \approx 7.7\, \rho n_R^{1/2} \approx 300\text{MeV}$[40]
Meson correlator	$\langle \bar\psi\Gamma\psi(x)\bar\psi\Gamma\psi(0)\rangle_{trunc.} \sim N_c$
Quark loop	 $\sim N_c$
Instanton scattering	 $\sim N_c^{-1} \rho n_R^{-1/2}$
Instanton occurrence	$\sim N_c n_R$
Gluon loop	 $\sim N_c^2 - 1$
Instanton scattering	 $\sim (N_c^2 - 1)^{-1}$

Table 3.1: Dependence of various quantities on the parameters of the instanton liquid model $n_R, \rho, N_c, (g_R)$.

This multiplies the graph (see table 3.1) by a factor loop $= O(N_c^{1+N-S})$. The following cases are possible:

$1 + N - S$	
$= 1$	$\Longleftrightarrow M = N \Longleftrightarrow$ all instantons are new
	$\Longleftrightarrow$ the loop is disconnected
$= 0$	$\Longleftrightarrow$ one old instanton $\Longleftrightarrow$ the loop is a tadpole
< 0	the loop is suppressed by at least one $1/N_c$

Disconnected graphs are cancelled by the denominator and tadpoles have been absorbed in n_R. So quark loops are indeed suppressed in the limit $N_c \to \infty$. The same is true for gluon loops. This can be seen by the same argument using the N_c dependencies from table 3.1.

3.10 Summary

The quark propagator has been computed in the Instanton Liquid Model in zeromode approximation and in $1/N_c$ expansion. It was possible to absorb dynamical quark loops into a renormalized instanton density, which should be identified with the gluon condensate. In the case of one quark flavor the $1/N_c$ expansion is exact. The values for the dynamical quark masses of the up, down and strange quark have been computed. Dynamical quark loops are suppressed in every order perturabtion theory in the limit $N_c \to \infty$. For the quark propagator this fact remains valid beyond perturbation theory. In the next chapter we will see that in meson correlators there are non-suppressed quark loops.

Chapter 4

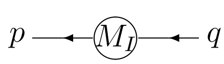
Four Point Functions

In this chapter the correlators for four quark fields in the Instanton Liquid Model in zeromode approximation will be computed. Of special interest is the observation that in leading order $1/N_c$ not all quark loops are suppressed. Summation of all graphs leads to Bethe Salpeter equations in Section 4.2, which are solved in Section 4.3. Section 4.4 summarizes the results for the connected and free flavor singlet and triplet correlators.

4.1 Introduction

In the last chapter we have derived comprehensive Feynman rules within the zero mode approximation:

$$= \frac{1}{\not p + im} = S_0(p)$$



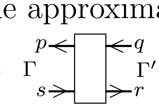
$$= \frac{1}{i\mu} \not p \psi_I(p) \psi_I^\dagger(q) \not q$$

A quark of a given flavor can scatter only once at a instanton I via M_I. Tadpole graphs are not allowed, they are absorbed in the renormalized instanton density n_R which has to be used when averaging over the instantons. In the large N_c limit dynamical quark loops are suppressed and μ can be determined by (3.19). In the chiral limit

$$\mu^2 = 6.6 n_R \rho^2 + O(n_R^2) \quad . \tag{4.1}$$

In this section we want to calculate 4 point functions in the case of two quark flavors of equal mass in the limit $N_c \to \infty$ within the zero mode approximation:

$$\bar\delta(p - s + r - q)\Pi_{\Gamma\Gamma'}(p, s, q, r) = \tag{4.2}$$

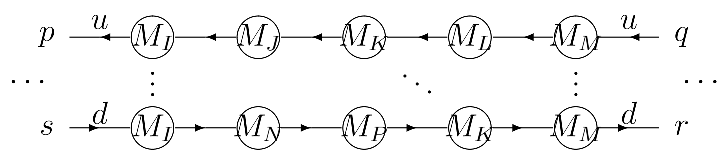


$$= - \int dxdydzdw\, e^{i(py-qz+rw-sx)} \langle 0| \mathcal{T} \bar\psi(x) \Gamma \psi(y) \bar\psi(z) \Gamma' \psi(w) |0\rangle \quad .$$

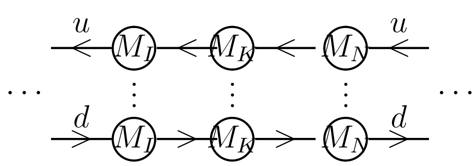
The ψ fields are u or d quark fields arbitrarily mixed. Without restriction to generality we have taken the correlator to be a color singlet . These 4 point functions can be used to study meson correlators (↑chapter 5) and quark form factors (↑chapter 6)

4.2 Large N_c approximation *

The most general graph for the quark propagator is a sequence of different instantons M_I, according to the rules above. Similarly the most general graph for the triplet correlator $\langle(\bar{u}d)(\bar{d}u)\rangle$ consists of two quark propagators each containing every insanton only once. However, the u and d propagator may contain common instantons, e.g.

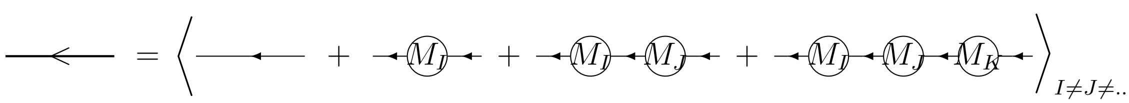


It is always assumed that the left and right hand sides of the graphs form color singlets. Non-common instantons can be averaged like in the propagator case to yield graphs of the form

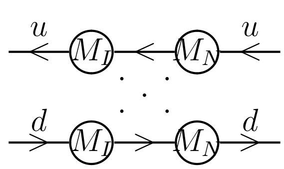


(4.3)

where the thick line represents the full propagator

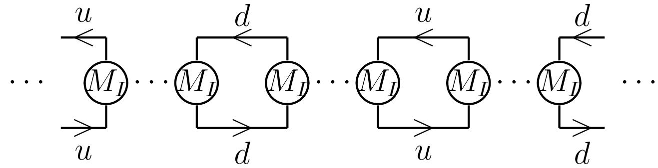


It can be shown that non planar diagrams like



are suppressed by $1/N_c$, where again the left and right hand sides of the graphs have to form color singlets. So only the ladder diagrams shown in (4.3) contribute to the triplet correlator.

One might think that the mixed correlator $\langle(\bar{u}u)(\bar{d}d)\rangle$ is zero because the graphs necessarily include quark loops, or that only the two loop graphs contribute — but this is not the case! Let us first state the result and then discuss it. Graphs contributing to the mixed correlator are chains of quark bubbles



(4.4)

Application of the N_c counting rules shows that this chain is of order $1/N_c$. Taking the color trace at the left and right hand side of the chain we see that the mixed correlator is of order N_c. Using (3.16) it is clear that the triplet correlator is of the same order.

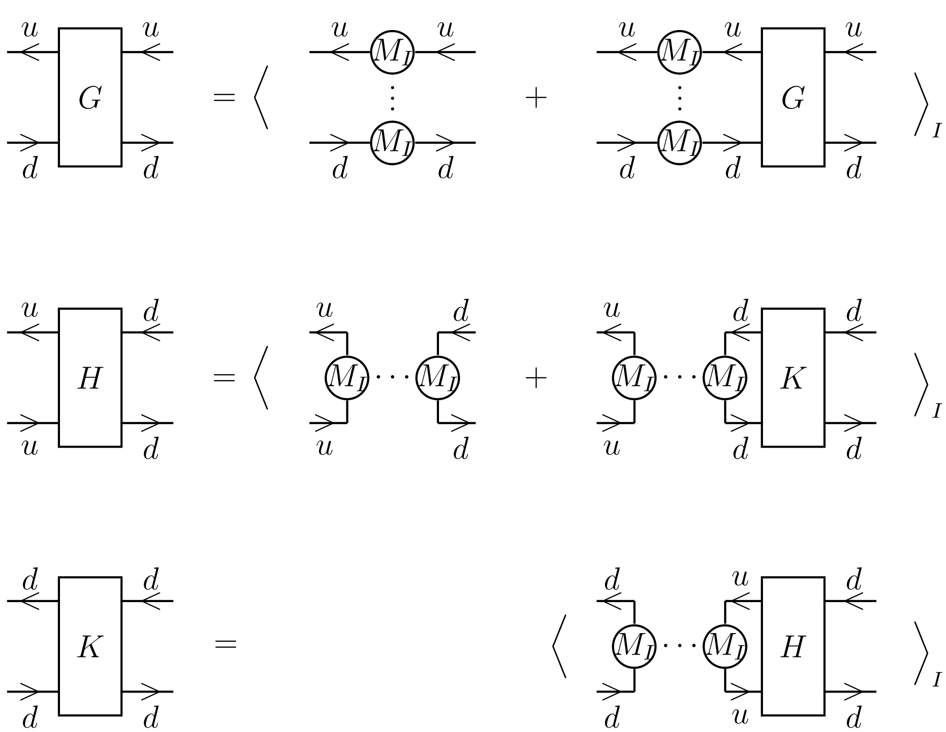
What is wrong with the derivation of quark loop suppression in the last chapter ? The main assumption was that every graph containing a loop can be constructed from a graph not possessing this loop by simply adding the loop. Eliminating a loop from the bubble chain (4.4) yields a disconnected graph, but we only consider connected 4 point functions. So the quark loop chain cannot be constructed in a way needed to prove quark loop suppression.

In the case of meson correlators one can take another point of view. The disconnected two loop contribution is of order N_c^2 but (except for the scalar case) the contribution is zero. So the bubble chain is a subleading graph of order N_c and nothing has been said about the form of subleading graphs.

Nevertheless, all connected graphs can be obtained starting from (4.4) by adding further instantons and bubbles — but now N_c counting rules tell us that every attempt results in a $1/N_c$ suppression. Therefore the bubble chain is the most general leading order graph.

Nothing has to be changed for the correlator $\langle(\bar{d}d)(\bar{d}d)\rangle$ except that chain (4.4) must start with d.

To calculate the connected 4 point functions one must now average and sum up the chains. Alternatively this can be represented in recursive from usually called Bethe Salpeter equations:

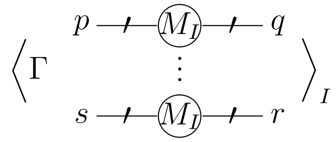


(4.5)

4.3 Solution of the Bethe-Salpeter Equations *

Before solving the BS equations we have to construct the kernel. The l.h.s. of the kernel always forms a color singlet because of the restriction to color singlet correlators. In the following we work in singular gauge suitable for small momentum (↑Chapter 8).

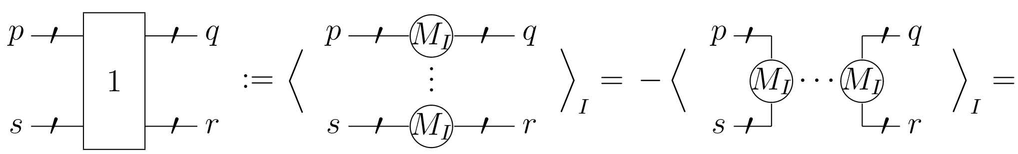
Contracting the color and Dirac indices on the l.h.s. and using the formulas of Appendices A.2 and A.3 one gets



$$= \frac{1}{(i\mu)^2}\langle \not r \psi_I(r)\psi_I^\dagger(s)\not s \Gamma \not p \psi_I(p)\psi_I^\dagger(q)\not q\rangle_I = \qquad (4.6)$$

$$= -\frac{n_R}{\mu^2} p\varphi'(p) q\varphi'(q) r\varphi'(r) s\varphi'(s)\hat\delta(p-s+r-q)\left\langle \mathrm{tr}_D\left(\Gamma\frac{1\pm\gamma_5}{2}\right)\frac{1\pm\gamma_5}{2}\right\rangle_\pm .$$

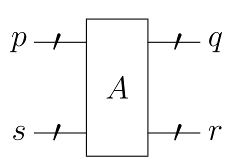
The kernel can now be determined to be



$$= -\frac{1}{n_R N_c}\sqrt{M_p M_q M_r M_s}\hat\delta(p-s+r-q)(\delta^{i_p}_{\ i_s}\delta^{i_r}_{\ i_q} + \gamma_5{}^{i_p}{}_{i_s}\gamma_5{}^{i_r}{}_{i_q})\delta^{\alpha_p}_{\alpha_s}\delta^{\alpha_r}_{\alpha_q} . \qquad (4.7)$$

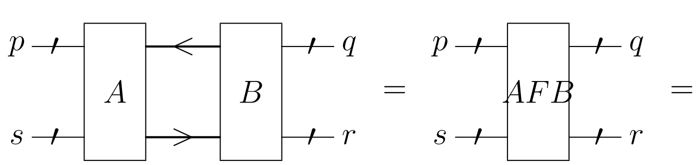
The result is just proportional to the nonlocal version of the 't Hooft vertex between color singlet states.

The solutions of the BS equations have a very similar structure:



$$:= -\frac{1}{n_R N_c}\sqrt{M_p M_q M_r M_s}\hat\delta(p-s+r-q) \qquad (4.8)$$
$$(A_0(t)\delta^{i_p}_{\ i_s}\delta^{i_r}_{\ i_q} + A_5(t)\gamma_5{}^{i_p}{}_{i_s}\gamma_5{}^{i_r}{}_{i_q})\delta^{\alpha_p}_{\alpha_s}\delta^{\alpha_r}_{\alpha_q} .$$

A_0 and A_5 are scalar functions depending only on $t = p - s = q - r$. The proof is simple: The kernel has the structure (4.8) with $A_0 = A_5 = 1$. The product of two vertices yields the same structure:



$$
\begin{aligned}
&= -\frac{1}{n_R N_c}\sqrt{M_p M_q M_r M_s}\,\bar{\delta}(p-s+r-q) \\
&\quad (A_0 F_0 B_0(t)\delta^{i_p}_{\ i_s}\delta^{i_r}_{\ i_q} + A_5 F_5 B_5(t)\gamma_5{}^{i_p}_{\ i_s}\gamma_5{}^{i_r}_{\ i_q})\delta^{\alpha_p}_{\ \alpha_s}\delta^{\alpha_r}_{\ \alpha_q} \quad ,
\end{aligned} \tag{4.9}
$$

$$
\begin{aligned}
F_0(t) &= -\int (dpds)\frac{1}{n_R}M_p M_s \mathrm{tr}_D(S(p)S(s)) \quad , \\
F_5(t) &= -\int (dpds)\frac{1}{n_R}M_p M_s \mathrm{tr}_D(S(p)\gamma_5 S(s)\gamma_5) \quad .
\end{aligned} \tag{4.10}
$$

$$
\int (dpds) := \int \bar{d}p\,\bar{d}s\,\bar{\delta}(p-s-t) \quad .
$$

In other words, the vertices of structure (4.8) build a closed algebra. The reason for this simple result is that the kernel is a simple product function up to the momentum conserving δ.

Using (4.7), (4.8) and (4.9) the BS equations (4.5) reduce to primitive algebraic equations for $G_{0/5}(t)$, $H_{0/5}(t)$ and $K_{0/5}(t)$:

$$
\begin{aligned}
G_{0/5}(t) &= 1 + F_{0/5}(t)G_{0/5}(t) \quad , \\
H_{0/5}(t) &= -1 - F_{0/5}(t)K_{0/5}(t) \quad , \\
K_{0/5}(t) &= -F_{0/5}(t)H_{0/5}(t) \quad ,
\end{aligned}
$$

with the solution

$$
G = \frac{1}{1-F} \quad , \quad H = -\frac{1}{1-F^2} \quad , \quad K = \frac{F}{1-F^2} \tag{4.11}
$$

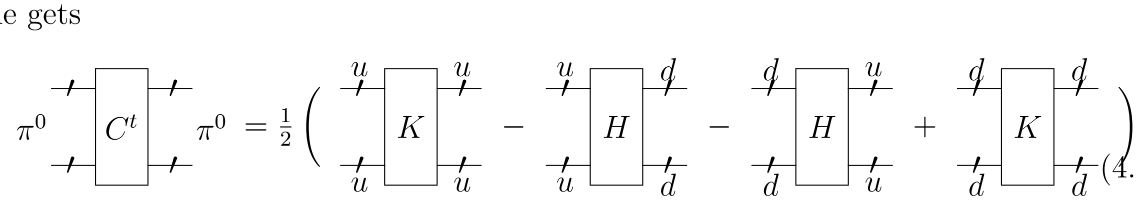
where we have suppressed the index 0/5 and the argument t.

4.4 Triplet and Singlet Correlators *

Because of isospin symmetry $SU(2)_f$, mesons form triplets and singlets. Replacing $\bar{\psi}\psi$ in (4.2) by the triplet and singlet combinations (borrowing the notation from the pseudoscalar correlator)

$$
\pi^0 = \frac{1}{\sqrt{2}}(\bar{u}u - \bar{d}d), \quad \pi^+ = \bar{u}d, \quad \pi^- = \bar{d}u, \quad \eta = \frac{1}{\sqrt{2}}(\bar{u}u + \bar{d}d), \tag{4.12}
$$

one gets



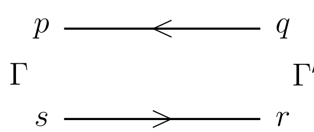
(4.13)

Therefore $C^t = K - H = \frac{1}{1-F}$. This coincides with $G = \frac{1}{1-F}$ for the charged triplet correlator $\langle(\pi^\pm)(\pi^\pm)\rangle$ as it should be. In the singlet case we get $C^s = K + H = -\frac{1}{1+F}$. Generalization to N_f flavors causes no problems. When adding propagators in (4.8) to the external legs the final result for the connected 4 point function is

$$\begin{aligned}\Pi^{conn}_{\Gamma\Gamma'}(p,s,q,r) &= -\frac{N_c}{n_R}\sqrt{M_pM_qM_rM_s}[C_0(t)\mathrm{tr}_D(S(s)\Gamma S(p))\mathrm{tr}_D(S(q)\Gamma S(r)) + \\ &\quad C_5(t)\mathrm{tr}_D(S(s)\Gamma S(p)\gamma_5)\mathrm{tr}_D(S(q)\Gamma S(r)\gamma_5)] \qquad (4.14)\end{aligned}$$

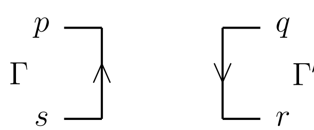
$$\begin{aligned}C^s_{0/5}(t) &= -\frac{N_f-1}{(N_f-1)F_{0/5}(t)+1} \quad \text{for the singlet correlator} \qquad (4.15)\\ C^t_{0/5}(t) &= -\frac{1}{F_{0/5}(t)\pm 1} \quad \text{for the triplet correlator}\end{aligned}$$

$F_{0/5}(t)$ are defined in (4.10). The correlators of a singlet with a triplet current are zero as expected. For completeness we give the expressions for the disconnected parts of the 4-point functions. The following graphs may contribute, depending on the flavor structure of the correlator:



$$= N_c\mathrm{tr}_D(\Gamma S(p)\Gamma'(s))\eth(p-q)\eth(r-s)$$

(4.16)



$$= -N_c^2\mathrm{tr}_D(\Gamma S(p))\mathrm{tr}_D(\Gamma'(q))\eth(p-s)\eth(q-r)$$

Note that the second two loop term is of the order N_c^2. However, as discussed above, in most applications it drops out or yields an uninteresting constant or only the connected part is considered anyway.

For the triplet and singlet case one gets

$$\eth(p-s+q-r)\Pi^{disc}_{\Gamma\Gamma'}(psqr) = N_c\mathrm{tr}_D(\Gamma S(p)\Gamma' S(s))\eth(p-q)\eth(r-s) + \begin{cases} 0 & \text{for triplet} \\ 2\cdot(4.16) & \text{for singlet}\end{cases} \qquad (4.17)$$

4.5 Summary

4 point correlators in the Instanton Liquid Model in zeromode approximation and $1/N_c$ expansion have been computed. The $1/N_c$ is the substitute for density expansion, which breaks down in the presence of light quarks. In contrast to the original "perturbation theoretic" $1/N_c$ expansion [62] not all quark loops are suppressed. In the flavor singlet channel a chain of quark loops survives the $N_c \to \infty$ limit. The obtained expressions for the 4 point functions will be discussed in the following chapter.

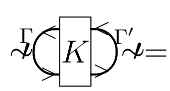
Chapter 5

Correlators of Light Mesons

Meson correlators, also called polarization functions, contain information about the meson spectrum of QCD. Poles at $p^2 = m^2$ show the existence of mesons with mass m. With the 4 point functions, calculated in the Instanton Liquid Model (↑Chapter 4), we also possess the meson correlators analytically, besides integration (Section 5.1). In Section 5.1 we will discuss them briefly, whereby the pseudoscalar channel is of special interest. Due to chiral symmetry breaking there are massless Goldstone bosons in the triplet channel (but not in the singlet channel), because instantons explicitly break the $U(1)_A$ symmetry. Both phenomenons have analogies in superconductivity. In order to compute the meson masses we make an ansatz for the meson spectrum in Section 5.3. Comparing the theoretical curve with this approach in Section 5.4 allows to determine the masses of the lightest mesons in the various channels. The evaluation of the integrals and the fit will be done numerically.

5.1 Analytical Expressions

In the last chapter we have calculated various quark 4 point functions (4.2). The meson correlators or polarization functions are just local versions of these vertices and can be obtained by simply setting $x = y$ and $z = w$. In momentum space the meson correlators have the form

$$\Pi_{\Gamma\Gamma'}(t) = \Pi^{disc}(t) + \Pi^{conn}(t) =$$ 

$$= \tag{5.1}$$

$$= \int (dpds) \int (dqdr) \Pi(p,s,q,r) = -\int dx\, e^{itx} \langle 0 | \mathcal{T} j_\Gamma(x) j_{\Gamma'}(0) | 0 \rangle \quad ,$$

$$j_\Gamma(x)^{s/t} = \frac{1}{\sqrt{2}}(\bar{u}\Gamma u(x) - \bar{d}\Gamma d(x)) \quad , \quad j_{\Gamma'}(0)^{s/t} = \frac{1}{\sqrt{2}}(\bar{u}\Gamma' u(0) - \bar{d}\Gamma' d(0)) \quad ,$$

$$\int (dpds) = \int \bar{d}p \bar{d}s\, \bar{\delta}(p - s - t) \quad , \quad \int (dqdr) = \int \bar{d}q \bar{d}r\, \bar{\delta}(q - r - t) \quad ,$$

$$t = p - s = q - r \quad .$$

From the explicit expressions of the 4 point functions obtained in the last chapter one can get, up to integration, analytical expressions for the meson correlators. The following list is a complete summary of all formulas needed to evaluate the meson correlators:

$$\begin{aligned}
\Pi^{disc}_{\Gamma\Gamma'}(t) &= N_c \int (dpds)\, \mathrm{tr}_D(\Gamma S(p)\Gamma' S(s)) \quad , \\
\Pi^{conn}_{\Gamma\Gamma'}(t) &= -N_c(C_0(t)\Gamma^0_\Gamma(t)\Gamma^0_{\Gamma'}(t) + C_5(t)\Gamma^5_\Gamma(t)\Gamma^5_{\Gamma'}(t)) \quad , \qquad (5.2)\\
C_{0/5}(t) &= -\frac{1}{F_{0/5}(t)\pm 1} \quad \begin{matrix} + \text{ for singlet correlator} \\ - \text{ for triplet correlator}\end{matrix} \quad , \\
\Gamma^0_\Gamma(t) &= \frac{1}{\sqrt{n_R}} \int (dpds)\, \sqrt{M_p M_s}\mathrm{tr}_D(S(p)\Gamma S(s)) \quad , \\
\Gamma^5_\Gamma(t) &= \frac{1}{\sqrt{n_R}} \int (dpds)\, \sqrt{M_p M_s}\mathrm{tr}_D(S(p)\Gamma S(s)\gamma_5) \quad , \\
F_0(t) &= \frac{-1}{n_R} \int (dpds)\, M_p M_s \mathrm{tr}_D(S(p)S(s)) \quad , \\
F_5(t) &= \frac{-1}{n_R} \int (dpds)\, M_p M_s \mathrm{tr}_D(S(p)\gamma_5 S(s)\gamma_5) \quad , \\
S(p) &= \frac{1}{\not p + i(m+M_p)} \quad , \quad M_p = \frac{n_R}{2\mu} p^2 \varphi'^2(p) \quad , \quad , \\
p\varphi'(p) &= 2\pi\rho z \frac{\partial}{\partial z}[I_0(z)K_0(z) - I_1(z)K_1(z)]_{z=p\rho/2} \quad , \\
\mu &= m + \int d\!\!{}^-\!{}^4 p\, \frac{2\varphi'^2(p) M_p}{p^2+(m+M_p)^2}(p^2 + m(m+M_p)) \quad , \qquad (5.3)\\
n_R N_c &= (200\mathrm{MeV})^4 \quad , \quad \rho = (600\mathrm{MeV})^{-1} \quad .
\end{aligned}$$

5.2 Analytical Results

Performing the Dirac traces leads to the following expressions:

$$\begin{aligned}
F_{0/5}(t) &= -\frac{4}{n_R} \int (dpds) \frac{M_p M_s(\pm(ps) - \tilde M_p \tilde M_s)}{(p^2+\tilde M_p^2)(s^2+\tilde M_s^2)} \quad , \\
\Gamma^{0/5}_{1/5}(t) &= \frac{4}{\sqrt{n_R}} \int (dpds) \frac{\sqrt{M_p M_s}(\pm(ps) - \tilde M_p \tilde M_s)}{(p^2+\tilde M_p^2)(s^2+\tilde M_s^2)} \quad , \qquad (5.4)\\
\Gamma^5_{\mu 5}(t) &= \frac{4i}{\sqrt{n_R}} \int (dpds) \frac{\sqrt{M_p M_s}(\tilde M_p s_\mu - \tilde M_s p_\mu)}{(p^2+\tilde M_p^2)(s^2+\tilde M_s^2)} \quad , \\
\tilde M_p &= m + M_p
\end{aligned}$$

All other vertices Γ are zero. Consider the one instanton vertex (4.7) (the kernel). It contributes only to the scalar and pseudoscalar correlator. From this observation one may have predicted that the connected part of all other channels is small because a contribution has to be a multi-instanton effect. Indeed, they are all zero as seen above except for the axial correlator. Due to an extra factor $M \sim \sqrt{n_R}$ in the numerator of $\Gamma^5_{\mu 5}$ the connected part of the axial correlator is suppressed by $O(n_R)$ therefore it is small as expected and will be neglected in the following.

Furthermore we will restrict ourself to the *chiral limit*, taking $m = 0$. Using the selfconsistency equation (5.3) one can see that

$$F_5(t=0) = 1 \tag{5.5}$$

is leading to a pole at $t = 0$ in the pseudoscalar triplet correlator due to the $F_5(t) - 1$ denominator in (5.2). This is the massless Goldstone pion one expects in the chiral limit. A more extensive discussion can be found in [25]. On the other side in the singlet correlator the minus sign is replaced by a positive sign and there is no Goldstone boson in this case. Thus, the (two flavor) η' meson is massive ! Unfortunately we cannot make any reliable prediction of the η' mass because the kernal is very repulsive in this channel and no boundstate is formed. There have to be other attractive forces, e.g. confinement forces, to built an η' boundstate. Similar things happen in the scalar triplet channel (compare figure B.5 and B.8). But the most important thing is that there is no massless pseudoscalar singlet meson which is an important step towards discussing the $U(1)_A$-problem.

Both phenomenons have their analogon in the BCS-theory of superconductors, namely massless excitons and massive plasma oscillations. The phenomenons as well as the structure of the calculation can be nearly 1:1 mapped onto, of course with differences in details. The 't Hooft vertex corresponds to the phonon-electron interaction. Further parallel can be read from table 5.1. An introduction to the BCS-theory can be found in [60].

It is interesting to see that in leading order in the instanton density $F_0(t) = -F_5(t)$, which leads to a massless pole in the scalar singlet correlator. Numerically the σ-meson indeed turns out to be very light. Numerically a light scalar singlet meson cannot be excluded.

5.3 Spectral Representation

To extract phenomenological information from the meson correlators we make use of the spectral representation

$$\Pi(p) = \int dx\, e^{ipx} \langle 0|\mathcal{T} j_\Gamma(x) j_{\Gamma'}(0)|0\rangle = \int_0^\infty d\sigma^2\, D(\sigma, x)\rho(\sigma^2) \tag{5.6}$$

where

$$\rho(p^2) = (2\pi)^3 \sum_n \delta(p - q_n)\langle 0|j_\Gamma(0)|n\rangle\langle n|j'_\Gamma(0)|0\rangle \tag{5.7}$$

is the spectral density and

$$D(m, x) = \int d\!\!{}^-{}^4p \frac{e^{-ipx}}{p^2 + m^2} = \frac{1}{4\pi^2 x^2}(mx)K_1(mx) \tag{5.8}$$

is the free propagator of mass m in coordinate representation. We have chosen the coordinate representation of Π to be able to compare the plots directly with lattice calculations and with numerical studies of the instanton liquid [28].

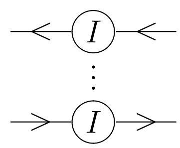
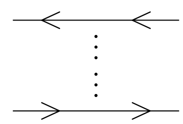
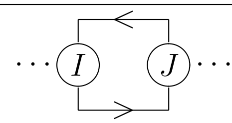
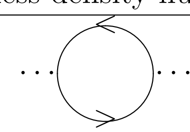
	Instantons in QCD	**BCS-Theory**
Phnomen	Chiral Symm.-breaking	Superconductivity
Boundstates ...	$\bar{q}q$-pairs=π-mesons	$e^- e^-$-pairs=Cooper-pairs
bounded by attractive ...	instanton-induced 't Hooft interaction 	phonon-electron-interaction 
in the ...	pseudo-scalar channel	scalar channel
Orderparameter	Quark condensate $\langle\bar{\psi}\psi\rangle$	Density of superconducting electrons (Cooper-pairs) $\langle\rho\rangle$
Spontaneous breaking of	chiral symmetry	gauge symmetry
leads to ...	Goldstone bosons = massless pions	Excitons = massless density fluct.
Vacuum polarization diagrams make the massless excitations massive	 in the flavor singlet case $\Longrightarrow$ massive η'	 for the Coulomb interaction $\Longrightarrow$ massive plasma osz.

Table 5.1: *Analogy between chiral symmetry breaking in QCD and the BCS theory of superconductors [60]*

Correlator		$\Gamma=\Gamma'$	I=1	I=0
Pseudoscalar	$\Pi_5 = \langle j_5 j_5 \rangle$	$i\gamma_5$	π	η'
Scalar	$\Pi_1 = \langle j_1 j_1 \rangle$	$\mathbb{1}_D$	δ	σ
Vector	$\Pi_{\mu\mu} = \langle j_\mu j_\mu \rangle$	γ_μ	ρ	ω
Axialvector	$\Pi^5_{\mu\mu} = \langle j^5_\mu j^5_\mu \rangle$	$\gamma_\mu\gamma_5$	a_1	f_1

Table 5.2: Mesonic correlators

The spectrum consists of mesonic resonances and the continuum contributions. If one is only interested in the properties of the first resonance one might approximate the rest of the spectrum by the perturbatively calculated continuum.

One might think that the disconnected part only contributes to the continuum and the connected part will yield the boundstates. But this is not the case. On one hand, Bethe Salpeter equations have bound as well as continuum solutions. On the other hand consider a theory with weak attraction between particles of mass m. It is clear that there is only a cut above $2m$ and no boundstate pole in the free loop. However in the exact polarization function only a small portion of the continuum will be used to form a pole just below the threshold because the attraction is only weak. The Euclidian correlator will hardly be changed. Therefore, assuming weak attraction, we can already estimate the boundstate mass from the disconnected part. Of course in this example we need not calculate anything because we know that the bounstate mass is approximately $2m$ with errors of the order of the strength of the interaction.

Assuming that all other forces neglected in QCD so far, especially perturbative corrections, are small and attractive in the vector and axialvector channel, we can obtain boundstate masses although in these channels up to our approximation there is no connected part. But things are less trivial than in the example above because the quarks do not posses a definite mass and we have to inspect the correlator to extract the meson masses.

Let us start with the scalar and pseudoscalar correlator. The lowest resonance of mass m_* is coupled to the current with strength

$$\lambda_* = \langle 0 | j_{1/5}(0) | p \rangle \quad . \tag{5.9}$$

The rest of the spectrum is approximated by the continuum starting at the threshold E_*. * means π, η, δ or σ (see table 5.2). E_* is typically of the order 1.5 GeV and therefore the continuum can be calculated perturbatively. The spectrum thus has the form

$$\rho_{1/5}(s) = \lambda_*^2 \delta(s - m_*^2) + \frac{3s}{8\pi^2}\Theta(s - E_*^2) \quad . \tag{5.10}$$

Inserting ρ into (5.6) one gets

$$\Pi^{fit}_{1/5}(x) = \lambda_*^2 D(m_*, x) + E(E_*, x) \quad , \tag{5.11}$$

$$E(E_*, x) = \frac{3}{\pi^4 x^6}\frac{(E_* x)^3}{16}(2K_3(E_* x) + (E_* x)K_2(E_* x)) \xrightarrow{x\to 0} \frac{3}{\pi^4 x^6} \quad . \tag{5.12}$$

In the next section m_*, λ_* and E_* are obtained by fitting the phenomenological ansatz $\Pi^{fit}(x)$ to the theoretical curve $\Pi^{sum}(x)$ in the Euclidian region where the theoretical calculation is reliable.

Consider now the vector and axial vector correlator. The vector current is conserved, thus the correlator is transverse and only the vector meson can contribute. In the chiral limit the same holds true for the axial current. In the singlet channel one has to be careful because there are two currents. A conserved one and a gauge invariant one which contains an anomaly. Up to now we have only calculated the correlator of the conserved current. Nevertheless to leading order in the instanton density the two correlators coincide and should be both conserved.

For conserved vector and axial currents the spectral function is transverse:

$$\rho^{(5)}_{\mu\nu}(p^2) = (-\delta_{\mu\nu} + \frac{p_\mu p_\nu}{p^2})\rho^{(5)}_T(p^2) \quad . \tag{5.13}$$

The coupling of the vector and axial meson to the current is given by

$$i\lambda_* \epsilon_\mu = \langle 0|j^{(5)}_\mu(0)|p\rangle \tag{5.14}$$

where ϵ_μ is the meson polarization. The spectral and polarization functions have the form

$$\begin{aligned} -\rho^{(5)}_{\mu\mu}(s) &= 3\lambda_*^2\delta(s-m_*^2) + \frac{3s}{4\pi^2}\Theta(s-E_*^2) \quad , \\ -\Pi^{fit(5)}_{\mu\mu}(x) &= 3\lambda_*^2 D(m_*,x) + 2E(x) \quad . \end{aligned} \tag{5.15}$$

Here * means ρ, ω, a_1 or f_1 (see table 5.2).

5.4 Plot & Fit of Meson Correlators

The meson correlators are shown in figure B.3 - B.8, normalized to the correlators in lowest order perturbation theory:

$$\Pi^0_{1/5}(x) = \frac{3}{\pi^4 x^6} \quad , \quad \Pi^{0(5)}_{\mu\mu} = -\frac{6}{\pi^4 x^6} \quad . \tag{5.16}$$

The diagrams therefore show the deviation from the perturbative behaviour. The numerical evaluation of the integrals are discussed in Appendices A.4 and A.5. The meson parameters obtained by fitting the parameter ansatz to the theoretical curve are summarized in table 5.3. Shuryak & Verbarshot [28] have obtained the same parameters from a numerical investigation of the instanton liquid model. Their values are also shown in table 5.3. The parameters of the vector channel coincide extremely well with ours. For the π and σ meson there is a large discrepancy in the masses but this is not surprising: We are working in the chiral limit thus the pion mass has to be zero. A similar argument holds for the σ meson as discussed above. The couplings fit very well.

The discrepancy in the axial channels can have various origins which are under investigation. Alternatively one may directly compare the graphs. They coincide very well even

Meson	$I^G(J^{PC})$	m_*[MeV]	$\sqrt{\lambda_*}$[MeV]	E_*[MeV]	source
π	$1^-(0^{-+})$	0	508±1	1276±33	$1/N_c$
		142±14	510±20	1360±100	simulation
		138	480	—	experiment
η'	$0^+(0^{-+})$	$\neq 0$	?	?	$1/N_c$
		$\neq 0$	?	?	simulation
		960	?	—	experiment
δ	$1^-(0^{++})$	$\neq 0$	?	?	$1/N_c$
		$\neq 0$	?	?	simulation
		970	?	—	experiment
σ	$0^+(0^{++})$	433±3	506±3	1446±20	$1/N_c$
		543	500	1160	simulation
		?	?	—	experiment
ρ	$1^+(1^{--})$	930±5	408±4	1455±33	$1/N_c$
		950±100	390±20	1500±100	simulation
		780	409±5	—	experiment
ω	$0^-(1^{--})$	930±5	408±4	1455±33	$1/N_c$
		?	?	?	simulation
		780	390±5	—	experiment
a_1	$1^-(1^{++})$	1350±200	370±30	1050±80	$1/N_c$
		1132±50	305±20	1100±50	simulation
		1260	400	—	experiment
f_1	$0^+(1^{++})$	1350±200	370±30	1050±80	$1/N_c$
		1210±50	293±20	1200±50	simulation
		1285	?	—	experiment

Table 5.3: Meson mass m_*, coupling constant λ_* and continuum threshold E_* obtained within the instanton liquid model in this work ($1/N_c$ expansion), from numerical simulation and from experiment.

in cases where a spectral fit does not work very well like in the δ and η' channel. The conclusion is that the terms neglected in our analytical treatment, but included in the numerical study [28], are small and usually give an correction less than 10%. These are contributions from nonzero modes and higher order corrections in $1/N_c$. This is again an example for the surprisingly high accuracy of the $1/N_c$ expansion. In the case of strange quarks the nonzero mode contributions will become more important.

Finally, one should compare the numbers with experiment. As far as known, these numbers are also listed in table 5.3. A general discussion of the meson correlators and comparison with experimental results can be found in [28].

Chapter 6

The Axial Anomaly

A variety of predictions concerning chiral symmetry breaking can be made within the instanton liquid model. Although the 't Hooft interaction [17] explicitly breaks the $U(1)$ axial symmetry, instanton models are up to now not too successful in describing quantitatively the axial singlet channel. The most interesting quantities are the η' mass and the spin of the proton.

In Section 6.1 we compute $m_{\eta'}$ by combining various techniques. The result does not depend on the specifics of instanton physics.

Sections 6.2, 6.3 and 6.4 are an introduction to the proton spin problem. In section 6.5 the proton form factors are reduced to vacuum correlators of 4 quark fields by assuming independent constituent quarks. The axial singlet quark and gluonic form factors are calculated in section 6.6, 6.7 and 6.8 by using the propagator and 4 point functions of the instanton liquid model. Gauge(in)dependence is examined. A discussion of the results and a comparison with [53] is given in section 6.9.

In Sections 6.2-6.5 we exceptionally use Minkowski-Notation.

6.1 The Mass of the η' Meson

In many channels a direct calculation of the meson correlators in the instanton liquid model and a spectral fit lead to reasonable results for the masses of the lightest mesons [28]. This method even works in the axial triplet channel because the model correctly describes spontaneous breaking of chiral symmetry. In the axial singlet channel a strong repulsion prevents the formation of a meson [26, 22]. The conclusion is, that there is no massless Goldstone boson in this channel, but the mass of the η' remains undetermined. In this letter I want to calculate the mass of the η' by combining quite different techniques. With the help of

- current algebra theorems for the η',
- $1/N_c$ expansion,

- instanton model,
- scale anomaly

we are able to relate the η' mass to the pion coupling constant f_π and the physical gluon condensate.

In leading order in $1/N_c$ it is possible to relate the η' mass to the Θ dependence of the topological susceptibility[1] $d^2E/d\Theta^2$ of QCD without quarks [43]:

$$m_{\eta'}^2 = \frac{4N_f}{f_\pi^2}\left(\frac{d^2E}{d\Theta^2}\right)_{\Theta=0}^{\textit{no quarks}} \quad , \quad \frac{d^2E}{d\Theta^2} = \int d^4x\, \langle 0|\mathcal{T}Q(x)Q(0)|0\rangle_{conn} \tag{6.1}$$

$$Q(x) = \frac{\alpha_s}{4\pi}\mathrm{tr}_c G\tilde{G}(x) \quad , \quad Q = \int d^4x\, Q(x) \in \mathbb{Z}$$

$Q(x)$ is the topological charge density and Q the total charge. This formula is derived by arguing, that for large N_c the topological susceptibility is dominated by the η' state, utilizing the axial anomaly [41] and the relation $f_\pi = f_{\eta'}$, which is exact for $N_c \to \infty$.

The next step is to relate the topological susceptibility to the gluon condensate $\langle 0|N(0)|0\rangle$:

$$N(x) = \frac{\alpha_s}{4\pi}\mathrm{tr}_c GG(x) \quad , \quad N = \int d^4x\, N(x)$$

In instanton models the gluon field consists of instantons of charge $Q = \pm 1$. The exact N instanton solutions ($Q = N$) are selfdual ($G_{\mu\nu} = \tilde{G}_{\mu\nu}$) and

$$\langle 0|\mathcal{T}Q(x)Q(0)|0\rangle_{conn} = \langle 0|\mathcal{T}N(x)N(0)|0\rangle_{conn} \quad . \tag{6.2}$$

The exact anti-instanton solutions ($Q = -N$) are anti-selfdual ($G_{\mu\nu} = -\tilde{G}_{\mu\nu}$) and (6.2) holds too, because the two minus signs cancel. Unfortunately these exact solutions are not the most important contributions to the partition function.

The dominating configurations are instantons and anti-instantons in mixed combination. The simplest model is a dilute sum $A = \sum_I A_I$ of instantons of mixed charge. $G_{\mu\nu}$ is then approximately selfdual near the instanton centers, approximately anti-selfdual near the anti-instanton centers and small far away from any instanton. In leading order in the instanton density we have

$$G_{\mu\nu}(x) = \pm\tilde{G}_{\mu\nu}(x) \tag{6.3}$$

where the sign now depends on x! Let us define $N^\pm(x)$ in the following way:

$$N(x) = N_+(x) + N_-(x) \quad , \quad Q(x) = N_+(x) - N_-(x)$$

$N_+(x)$ is a sum of bumps near the centers of instantons and $N_-(x)$ near the centers of anti-instantons. (6.2) has to be replaced by the relation

$$\langle 0|\mathcal{T}Q(x)Q(0)|0\rangle_{conn} = \langle 0|\mathcal{T}N(x)N(0)|0\rangle_{conn} - 4\langle 0|\mathcal{T}N_+(x)N_-(0)|0\rangle_{conn} \tag{6.4}$$

[1] $\langle AB\rangle_{conn} = \langle AB\rangle - \langle A\rangle\langle B\rangle$

Assuming independence of instantons and anti-instantons ($\langle N_+ N_- \rangle = \langle N_+ \rangle \langle N_- \rangle$) the equation reduces again to (6.2). We will see that this is a crucial assumption.

The next ingredient is the scaling behaviour of QCD. Classical chromodynamics is scale invariant and the Noether theorem leads to a conserved scale current. In quantum theory the scale invariance is anomalously broken (like the axial singlet current). Ward identities can be derived, especially [64]

$$\int d^4x \ \langle 0|\mathcal{T} N(x) N(0)|0\rangle_{conn} = \frac{4}{b}\langle 0|N(0)|0\rangle \quad , \quad b = \frac{11}{3} N_c \tag{6.5}$$

Therefore in a self(anti)dual background the topological susceptibility $d^2E/d\Theta^2$ is proportional to the gluon condensate:

$$\frac{d^2E}{d\Theta^2} = \frac{4}{b}\langle 0|N(0)|0\rangle \tag{6.6}$$

The relation is still valid, when there are statistically independent regions of selfduality and selfantiduality, as discussed above. It is in fact sufficient to assure independence of the total instanton/anti-instanton number $N_\pm$. I have checked (6.6) by using the theoretical one loop instanton density $D(\rho)$ calculated in [17]. Only the ρ dependence $D(\rho) \sim \rho^{-5}(\rho\Lambda)^b$ is important. Due to the infrared divergence it is necessary to introduce an infrared cutoff, but one has to assure not to break scale invariance. A minimal change is to introduce two cutoffs $f_\pm$ in the total instanton/anti-instanton packing fraction. The packing fraction is the spacetime volume occupied by the instantons and is a dimensionless quantity. Scale invariance and independence of instantons and anti-instantons are ensured. The partition function Z is

$$Z = \sum_{N_+ N_-} Z^+_{N_+} Z^-_{N_-} \tag{6.7}$$

$$Z^\pm_{N_\pm} = \frac{V_4^{N_\pm}}{N_\pm!}\int_0^\infty d\rho_1 \dots d\rho_{N_\pm}\, D(\rho_1)\dots D(\rho_{N_\pm})\,\Theta\left(f_\pm - \frac{1}{V_4}\sum_{i=1}^{N_\pm}\rho_i^4\right)$$

A lengthy, but quite standard calculation of statistical physics, leads to

$$Z^\pm_{N_\pm} = \left(\frac{c_\pm N_\pm}{V_4 \Lambda^4}\right)^{-\frac{bN_\pm}{4}}$$

where $c_\pm = c_\pm(f_\pm, b)$ are constants independent of $N_\pm$. Differentiation of $\ln Z$ w.r.t. $\ln c_\pm$ two times leads to (6.6). The result is independent of $f_\pm$. An attractive interaction would lower the susceptibility (compared to the density). This can be seen in the following way: In the extreme case of a very attractive interaction, all instantons will be bound to instanton-anti-instanton molecules, thus $N_+ = N_-$ and $Q \equiv 0$. On the other hand a repulsive interaction would increase the susceptibility:

$$\frac{d^2E}{d\Theta^2} < \frac{4}{b}\langle 0|N(0)|0\rangle \quad \text{for attractive } I\bar{I} \text{ interaction} \tag{6.8}$$

$$\frac{d^2E}{d\Theta^2} > \frac{4}{b}\langle 0|N(0)|0\rangle \quad \text{for repulsive } I\bar{I} \text{ interaction} \tag{6.9}$$

Therefore the violation of (6.6) is a measure of the $I\bar{I}$ interaction.

All this should be compared to Dyakonov [25], where the relation

$$d^2E/d\Theta^2 = \langle 0|N(0)|0\rangle \tag{6.10}$$

has been derived, which is similar to (6.6). In this work a simple sum ansatz $A = \sum A_I$ has been made. This ansatz leads to a strong repulsion between close instantons, which is the origin of the missing factor $4/b$. The result on its own and the comparison with (6.9) shows that some repulsive interaction is at work. Verbaarshot [20] has shown that this repulsion is an artifact of the simple sum ansatz. Using the much more accurate and elaborate streamline ansatz he showed, that the interaction strongly depends on the orientation and the average interaction is about 14 times smaller than those obtained in [25]. Therefore the best thing one can do today is to assume no interaction at all and cut the packing fraction at some small value. There is also a more general argument that the relation derived in [25] must be wrong. The topological susceptibility is of $O(N_c^0)$, whereas the gluon condensate is of $O(N_c)$. (6.6) is consistent with $N_c \to \infty$ considerations only due to the presence of the $4/b$ factor.

Let us now continue with the calculation of $m_{\eta'}$. The physical gluon condensate in the presence of light quarks is

$$\langle 0|N(0)|0\rangle_{phys} = (200\text{MeV})^4 \quad . \tag{6.11}$$

Due to the presence of light quarks it is reduced by a factor $\alpha < 1$.

$$\langle 0|N(0)|0\rangle_{phys} = \alpha \langle 0|N(0)|0\rangle_{\Theta=0}^{no\ quarks} \tag{6.12}$$

Combining (6.1), (6.2), (6.5) and (6.12), we get the final formula for the η' mass is

$$m_{\eta'}^2 = \frac{4N_f}{f_\pi^2}\frac{12}{11N_c\alpha}n_R \tag{6.13}$$

One can see that $m_{\eta'}^2 \sim 1/N_c$ because f_π^2 and n_R and α are independent of N_c. The largest uncertainty lies in the determination of α. Using again the instanton model, α is the determinant of the Dirac operator with the current quark masses replaced by effective masses:

$$\alpha = \prod_{i=u,d,s} 1.34 m_i^{eff}\rho \approx 0.4\ldots0.7 \tag{6.14}$$

We have set the effective masses to the constituent masses

$$m_u^{eff} = m_d^{eff} = 300\ldots350 \text{ MeV} \quad , \quad m_s^{eff} = 400\ldots500 \text{ MeV} \tag{6.15}$$

and ρ to the value of the instanton liquid model ($\rho = 600\ \text{MeV}^{-1}$). This estimate is consistent with the estimate of [21]. Inserting $N_f = 3$ and $f_\pi = 132$ MeV into (6.13) we get

$$m_{\eta'} = 884 \pm 116\text{MeV} \tag{6.16}$$

which is in good agreement to the experimental value of 958 MeV. This result in turn confirms the assumption, that the interaction between selfdual and anti-selfdual regions is small. The large uncertainty in α prevents more accurate statements, but (6.10) can definitely be excluded.

Using $m_{\eta'}$ as an input we can determine the gluon condensate in pure QCD

$$\frac{\alpha_s}{4\pi}\langle \mathrm{tr}_c GG\rangle^{no\ quarks} = (246\ \mathrm{MeV})^4 \tag{6.17}$$

where we have again set N_f to 3.

The factor $4/b$ in (6.6) is the essential term to get the correct N_c dependence of $m_{\eta'}$ and agreement with the experimental mass. The discussion has shown, that the η' channel can be an experimental device for testing the independence of selfdual and antiselfdual regions in QCD, which is an assumption in the simplest instanton models. It might turn out some day that the details of instanton models are wrong but the assumption of independent self(anti)dual regions remain valid.

6.2 Measurement of the Axial Form Factors

The forward matrix elements of the axial currents

$$s_\mu \Delta\psi = \langle ps|\bar{\psi}\gamma_\mu\gamma_5\psi|ps\rangle \quad , \quad \psi = u, d, s$$

can be interpreted as the quark spin content of the proton, in a sense defined more accurately in the following sections. Three independent linear combinations of $\Delta\psi$ have been measured, thus allowing to extract their individual values.

From the neutron β-decay, using isospin invariance, one gets [50, 54]

$$a_3 = g_A = \Delta u - \Delta d = F + D = 1.254 \pm 0.06 \quad .$$

From the octet hyperon β-decay, using $SU(3)_F$ symmetry, one gets [51, 54]

$$\sqrt{3}a_8 = \Delta u + \Delta d - 2\Delta s = 3F - D = 0.688 \pm 0.0035 \quad .$$

From the spin dependent structure function g_1^p of the proton, which has been measured by EMC [48] and SMC [49], one can extract

$$\Gamma_p = \int_0^1 g_1^p(x)dx = \frac{4}{9}\Delta u + \frac{1}{9}\Delta d + \frac{1}{9}\Delta s + O(\alpha_s) = 0.142 \pm 0.014 \tag{6.18}$$

where we have given the world average value.

Of special interest is the quarkspin sum, which can be extracted from the values given above,

$$\Delta\Sigma^{GI} = \sqrt{\frac{3}{2}}a_0 = \Delta u + \Delta d + \Delta s = 0.27 \pm 0.13 \tag{6.19}$$

where the $O(\alpha_s)$ corrections have been included. It deviates significantly from the naive quark model value $\Delta\Sigma_{qm} = 1$. This deviation is the origin of the so called spin problem. Further the large polarization of strange quarks in the proton

$$\Delta s = -0.1 \pm 0.05$$

is counter intuitive because this indicates a large strange quark content of the proton.

Much more could be said about proton spin phenomenology and the experiments. For an introduction and further references see [45, 46, 47, 48]. We will now give a more thorough definition and interpretation of $\Delta\Sigma^{GI}$ and other quantities, which we want to calculate within the instanton model.

6.3 Axial Singlet Currents & Anomaly

It is well known that products of operators at the same spacetime point are very singular objects. In order to make the expressions well defined one has to regularize and renormalize the operator products. An anomaly appears, if this procedure breaks a symmetry of the theory. The most important ones are the breakdown of the scale invariance and the breakdown of the axial symmetry [64]. In the following we are interested in the axial anomaly [41]. The operator product which has to be regularized is the axial singlet current,

$$J_{\mu 5}(x) = \sum_{q\in\{u,d,s,\dots\}} \bar{q}(x)\gamma_\mu\gamma_5 q(x) \tag{6.20}$$

which seems to be local, gauge invariant and conserved[2]. Unfortunately after regularization one of the three properties is unavoidably lost. Therefore we can define two different local currents, a conserved (c) one and a gauge invariant (GI) one. The third GI, conserved and non-local current is discussed in [42] in connection with the $U(1)$ problem. We will suppress the summation over quark flavors and write ψ for the quark field operator:

$$\begin{aligned} J^{GI}_{\mu 5}(x) &= \lim_{\varepsilon\to 0} \bar{\psi}(x+\varepsilon)\gamma_\mu\gamma_5 P\exp\left(i\int_x^{x+\varepsilon} dz\cdot A(z)\right)\psi(x) \\ J^{c}_{\mu 5}(x) &= \lim_{\varepsilon\to 0} \bar{\psi}(x+\varepsilon)\gamma_\mu\gamma_5\psi(x) \end{aligned} \tag{6.21}$$

The difference between the two currents is described by the anomaly current K_μ:

$$\begin{aligned} K_\mu(x) &= \frac{N_f\alpha_s}{2\pi}\varepsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c A^\nu(G^{\rho\sigma} - \frac{2}{3}A^\rho A^\sigma) \quad , \quad J^{GI}_{\mu 5} = J^c_{\mu 5} + K_\mu \\ \partial^\mu K_\mu(x) &= \frac{N_f\alpha_s}{2\pi}\mathrm{tr}_c G\tilde{G}(x) = a(x) \\ \partial^\mu J^c_{\mu 5}(x) &= 2mJ_5(x) \qquad , \quad J_5 = i\bar{\psi}\gamma_5\psi \quad . \end{aligned} \tag{6.22}$$

m is the current quark mass and N_f is the number of quark flavors. Note, that the splitting of $J_{\mu 5}$ in a conserved and an anomaly part is gauge dependent. There are attempts to

[2] We will use the term 'conserved' even for $m_q \neq 0$. Sometimes this current is called the symmetric current in the literature.

define both uniquely on physical grounds [46]. The intention is to define $J^c_{\mu 5}$ as the naive parton model spin and K_μ as some gluonic contribution. The proton matrix elements of the various currents can be expressed in terms of real form factors $G_i, K_i, AandJ$:

$$\begin{aligned}
\langle p's'|J^{GI}_{\mu 5}(0)|ps\rangle &= \bar{u}_{s'}(p')\left[\gamma_\mu\gamma_5 G^{GI}_1(q^2) - q_\mu\gamma_5 G^{GI}_2(q^2)\right] u_s(p) \\
\langle p's'|J^{c}_{\mu 5}(0)|ps\rangle &= \bar{u}_{s'}(p')\left[\gamma_\mu\gamma_5 G^{c}_1(q^2) - q_\mu\gamma_5 G^{c}_2(q^2)\right] u_s(p) \\
\langle p's'|K_\mu(0)|ps\rangle &= \bar{u}_{s'}(p')\left[\gamma_\mu\gamma_5 K_1(q^2) - q_\mu\gamma_5 K_2(q^2)\right] u_s(p) \\
\langle p's'|a(0)|ps\rangle &= 2MiA(q^2)\bar{u}_{s'}(p')\gamma_5 u_s(p) \\
\langle p's'|J_5(0)|ps\rangle &= iJ(q^2)\bar{u}_{s'}(p')\gamma_5 u_s(p)
\end{aligned} \tag{6.23}$$

M is the proton mass and $q = p' - p$. From (6.22) one can derive the following relations between the form factors:

$$\begin{aligned}
G^{GI}_1 = G^c_1 + K_1 \quad &, \quad G^{GI}_2 = G^c_2 + K_2 \\
G^c_1 - \frac{q^2}{2M}G^c_2 = \frac{m}{M}J \quad &, \quad K_1 - \frac{q^2}{2M}K_2 = A \\
G^{GI}_1 - \frac{q^2}{2M}G^{GI}_2 &= \frac{m}{M}J + A \quad ,
\end{aligned} \tag{6.24}$$

where all form factors are evaluated at q. The last equation relates only GI quantities. In the next section we show that the form factor G^{GI}_1 at zero momentum transfer can be connected with the proton spin.

6.4 The Proton Spin and its Interpretation

The stress tensor $T_{\mu\nu}$ is conserved ($\partial_\mu T^{\mu\nu} = 0$) symmetric and GI and can be constructed from the Noether theorem. The angular momentum density tensor $M^{\mu\nu\rho}$ associated with Lorentz transformations can be expressed in terms of $T_{\mu\nu}$:

$$M^{\mu\nu\rho} = x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu} \tag{6.25}$$

M can be decomposed in spin and orbital contribution of quarks and gluons [54]:

$$\begin{aligned}
M^{\mu\nu\rho} &= M^{\mu\nu\rho}_{q,orb} + M^{\mu\nu\rho}_{q,spin} + M^{\mu\nu\rho}_{g,orb} + M^{\mu\nu\rho}_{g,spin} - \frac{1}{4}G^2(x^\nu g^{\mu\rho} - x^\rho g^{\mu\nu}) + \partial(\cdots) \\
M^{\mu\nu\rho}_{q,orb} &= \frac{1}{2}i\bar{\psi}\gamma^\mu(x^\nu\partial^\rho - x^\rho\partial^\nu)\psi \quad , \quad M^{\mu\nu\rho}_{q,spin} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}\gamma_\sigma\gamma_5\psi = \frac{1}{2}J^{GI}_{\sigma 5} \\
M^{\mu\nu\rho}_{g,orb} &= -G^{\mu\sigma}(x^\nu\partial^\rho - x^\rho\partial^\nu)A_\sigma \quad , \quad M^{\mu\nu\rho}_{g,spin} = G^{\mu\rho}A^\nu - G^{\mu\nu}A^\rho
\end{aligned} \tag{6.26}$$

The last two terms in $M^{\mu\nu\rho}$ do not contribute to the angular momentum operator

$$J^i = \frac{1}{2}\varepsilon^{ijk}\int d^3x \; M^{0jk}(x) \quad . \tag{6.27}$$

Taking the matrix element of J_z in a proton state, where the proton is aligned in z-direction and at rest we get the spin of the proton

$$\Delta J = \frac{1}{\mathcal{N}} \langle ps|J_z|ps\rangle = \frac{1}{2}\varepsilon^{3jk}\langle ps|M^{0jk}(0)|ps\rangle \quad , \quad \mathcal{N} = \langle p,s|p,s\rangle = \delta^3(0) \tag{6.28}$$

The total spin of the proton is with no doubt 1/2 and we get the sum rule

$$\Delta J = \Delta L_q + \frac{1}{2}\Delta\Sigma^{GI} + \Delta L_g + \Delta g = \frac{1}{2} \tag{6.29}$$

where $(\Delta L_q, \frac{1}{2}\Delta\Sigma^{GI}, \Delta L_g, \Delta g)$ are the (quark-orbital, quark-spin, gluon-orbital, gluon-spin) contribution to the proton spin, defined as matrix elements of the various parts of M given above. Therefore The GI axial current measures the quark spin contribution to the proton spin. The space integral in 6.27 can be cancelt with the state normalization and we get in covariant notation:

$$s_\mu \Delta\Sigma^{GI} = \langle ps|J^{GI}_{\mu 5}(0)|ps\rangle \quad \Longrightarrow \quad \Delta\Sigma^{GI} = G^{GI}_1(0) \tag{6.30}$$

In the naive quark model the proton consists of three quarks at rest. There is no orbital and no gluonic contribution to the proton spin. This leads to the Ellis-Jaffe sum rule $\Delta J = \frac{1}{2}\Delta\Sigma^{GI} = 1/2$. In the real world the identification of $\Delta\Sigma^{GI}$ with the proton spin is not correct, because $J^{GI}_{\mu 5}$ measures the spin of the (nearly massless) current quarks whereas the proton consists of three massive (≈ 300 MeV) constituent quarks. Further in a model of non-interacting constituent quarks the axial current which measures the constituent quark spin should be anomaly free because the anomaly is due to the interaction with gluons. Therefore the conserved current $J^c_{\mu 5}$ might be identified with the constituent quark spin operator.

$$s_\mu \Delta\Sigma^c = \langle ps|J^c_{\mu 5}(0)|ps\rangle \quad \Longrightarrow \quad \Delta\Sigma^c = G^c_1(0) \stackrel{?}{=} 1 \quad . \tag{6.31}$$

From (6.24) we get

$$\Delta\Sigma^{GI} = \Delta\Sigma^c + K_1(0) \tag{6.32}$$

which can now be interpreted in the following way: The spin of the constituent quarks $\Delta\Sigma^c$ are formed by the spin of the current quarks $\Delta\Sigma^{GI}$ and a rest $-K_1(0)$, which contains orbital and gluonic contributions. The origin of these contributions is *not* the motion and interaction of the constituent quarks inside the proton, because the constituent quarks are noninteracting and at rest in the naive quark model, but due to the formation of massive quarks from massless quarks. Therefore (6.32) may be discussed for an individual "constituent" quark. Further the gluonic configurations which are responsible for the generation of the quarkmass also determine the value of $K_1(0)$.

E.g. in a BAG model a massive quark is formed by confining a massless quark to a sphere. The spin of the massive constituent quark is the sum of the spin $(\frac{1}{2}\Delta\Sigma^{GI})$ and the oribtal $\frac{1}{2}(1 - \Delta\Sigma^{GI})$ contribution of the current quark. The BAG, which might be formed by nonperturbative gluonic configurations, is responsible for the mass generation and indirectly for the orbital contribution. From analytical and numerical calculations we know, that in the BAG model the constituent spin is splitted into 70% spin and 30% orbital contribution when starting with massless quarks.

Whereas (6.32) is rigorously true, the interpretation of $\frac{1}{2}\Delta\Sigma^c$ as the spin of a constituent quark and its value $\frac{1}{2}$ is questionable. One reason is, that an axial current which describes massive constituent quarks is by no means conserved in contradiction to $J^c_{\mu 5}$.

There exists another relation between $\Delta\Sigma^{GI}$ and the form factor A at zero momentum transfer. Before deriving this relation we have to give a short discussion about the order of limits and massless poles. The following limits are taken: The spacetime volume goes to infinity ($V_4 \to \infty$), because the universe is actually very large, the current quark masses go to zero ($m \to 0$), because the up and down masses are very small and $q \to 0$, because we are interested in the forward matrix elements. In principle the results can depend on the order of the limits and therefore they have to be chosen consistent with the physical situation. This means, that if in the real world e.g. $q \ll m$ we first have to take $q \to 0$ and then $m \to 0$. Actually we are interested in the forward matrix element ($q \equiv 0$) and $m \neq 0$ in the real world and the order of limits just stated applies. Through the cluster theorem connected correlators in coordinate space have to decay to zero when the separation of two arguments tends to infinity. Therefore there are no $\delta(q)$-peaks in momentum space and the order of limits $q \to 0$ and $V_4 \to \infty$ can be taken at will. Because $m^4 V_4 \gg 1$ we have to take first $V_4 \to \infty$ and then $m \to 0$. In statistical physics this is a well known fact, that a spontaneous breakdown of a symmetry only occurs, when there is a small explicit symmetry breaking term and the system volume tends to infinity. In the final end one may remove the symmetry breaking term. In QCD chiral symmetry is spontaneously broken (SBCS) and the small current quark mass is the explicit breaking of the chiral symmetry. Therefore it is mandatory first to take $V_4 \to \infty$ and then $m \to 0$ [65]. Therefore we can use the following order of limits

$$\lim_{m\to 0}\{\lim_{q\to 0}[\lim_{V_4\to\infty}(\ldots)]\}. \tag{6.33}$$

This justifies the usage of the infinite volume formulation from the very beginning.

In real QCD there are no massless particles ($m_\pi \neq 0$). Therefore GI form factors have no massless poles especially

$$q^2 G_2^{GI}(q^2) \stackrel{q\to 0}{\longrightarrow} 0 \tag{6.34}$$

From (6.24), (6.30) and (6.34) we get

$$\Delta\Sigma^{GI} = \frac{m}{M} J(0) + A(0) \tag{6.35}$$

This relation is true whether there are Goldstone bosons in the axial singlet channel or not. Experimentally we know that the lightest particle in this channel is the η' with a mass of 958 MeV much too large to be a Goldstone boson. Therefore $J(0)$ remains finite in the chiral limit and we obtain

$$\Delta\Sigma^{GI} = A(0) \quad \text{for} \quad m \to 0 \tag{6.36}$$

Assuming the non-existence of the axial singlet Goldstone boson from the very beginning the order of limits is of no importance in deriving (6.36). Note, that (6.36) is only true, if

	$\Delta\Sigma^c = \frac{q^2}{2M}G_2^c + \frac{m}{M}J$	$K_1 = \frac{q^2}{2M}K_2 + A$	$\Delta\Sigma^{GI} - \frac{q^2}{2M}G_2^{GI} = \Delta\Sigma^c + K_1 = \frac{m}{M}J + A$
$N_f m = M$	1 = 0 + 1	0 = 0 + 0	1 - 0 = 1 + 0 = 1 + 0
$m = 0$	1 = 1 + 0	A-1= -1 + A	A - 0 = 1 +A-1= 0 + A
Instanton	? = ? + 0	0 = 1 +(-1)	1 - 0 = ? + 0 = 0 +(-1)

Table 6.1: *The proton form factors at zero momentum transfer $q^2 = 0$ in the naive constituent quark model ($N_f m = M$), in chiral QCD (m=0) and in the instanton-liquid model (Instanton). Experimentally A is 0.27.*

we take $m = m_u = m_d = m_s$, although all three masses tend to zero. Otherwise additional nonsinglet currents on the r.h.s. of (6.35) would survive the chiral limit [55].

Combining (6.24), (6.34) and (6.32) we can conclude that

$$2M\Delta\Sigma^c = q^2 G_2^c(q^2)_{|q^2=0} = -q^2 K_2(q^2)_{|q^2=0}$$

$\Delta\Sigma^c$ is given by the pole residuum of G_2^c. Because G_2^{GI} has no massless pole $\Delta\Sigma^c$ is also given by the pole of $-K_2$. These massless poles are called ghost poles and they may truly appear, even if there are no physical massless particles, because G_2^c and K_2 are gauge dependent objects. Note that all other form factors defined in (6.24) are GI and therefore free of massless poles.

Table 6.4 summarizes the values for the form factors at zero momentum transfer for the following three cases:

- the naive quark model of non-interacting constituent quarks of mass $m = M/N_f$,
- chiral QCD and the identification of $\Delta\Sigma^c$ with the naive spin value 1,
- the instanton liquid model.

In the following sections we will calculate some of the form factors for a single constituent quark in the instanton liquid model.

6.5 Reduction of the Proton Form Factors to Vacuum Correlators

In this section we will calculate some of the form factors defined above in the instanton liquid model. To apply the methods developed in [26] we relate the form factors to vacuum correlation functions

$$\langle p's'|B(0)|ps\rangle = \tag{6.37}$$

$$= -\frac{1}{Z_\eta}\bar{u}_{s'}(p')\left[\int d^4x\, d^4z\, e^{ip'x-ipz}(i\partial\!\!\!/_x - M)(-i\partial\!\!\!/_z - M)\langle 0|\mathcal{T}\eta(x)B(0)\bar{\eta}(z)|0\rangle\right]u_s(p) \quad .$$

M is the proton mass and $B(0)$ is an arbitrary local operator. $\eta(x)$ is a local operator with the quantum numbers of a proton e.g. a product of three quark fields in an appropriate spin and flavor combination [44]. Assuming that $\eta(x)$ tends to a free proton field operator for infinite times the proton states can be reduced and (6.37) is just an LSZ reduction formula for composite fields. For our purpose the following form is more suitable

$$\begin{aligned} \langle p's'|B(0)|ps\rangle &= Z_\eta \bar{u}_{s'}(p')[\lim_{p^2,p'^2\to M^2} S^{-1}(p')T_B(p',p)S^{-1}(p)]u_s(p) \\ T_B(p',p) &= \int d^4x\, d^4z\; e^{ip'x-ipz}\langle 0|\mathcal{T}\eta(x)B(0)\bar{\eta}(z)|0\rangle \\ S(p) &= \int d^4x\; e^{ipx}\langle 0|\mathcal{T}\eta(x)\bar{\eta}(0)|0\rangle = \frac{iZ_\eta}{\not{p}-M} + continuum \\ Z_\eta^{1/2}u_s(p) &= \langle 0|\eta(0)|ps\rangle \end{aligned} \tag{6.38}$$

The advantage of this form is, that the explicit knowledge of the mass M is not needed. In Euclidian calculations like lattice-, instanton- and OPE-calculations it is always difficult to extract pole masses.

This form can also be interpreted as a spectral representation of the 3 point function. Inserting two complete sets of states into the 3 point function and taking the limit $p^2 = p'^2 \to M^2$ to select the proton state one can directly attain (6.38).

If e.g. $B(0)$ is a quark current, the 3 point function is a product of 8 quark fields, which is too complicated to be evaluated in a multi-instanton background. Let us assume that the proton consists of three nearly independent quarks. Then the main nonperturbative properties of the proton come from the formation of constituent quarks out of current quarks. The forces which confine the constituent quarks in the proton are assumed to modify the properties of the proton only in a minor way, except that the proton is then stable. This assumption is justified by the success of the constituent quark model. The form factors of the proton are therefore the sum of the form factors of the constituent quarks. η has to be replaced by a single quark field ψ of flavor *up* or *down* and M must be replaced by the constituent quark mass. In this case it is even more important to use (6.38) because one does not expect a definite pole mass for the quark propagator. Looking at the quark propagator in the instanton liquid model we see, that the $\not{p}$ term remains unrenormalized and therefore $Z_\psi = 1$. For a constant constituent mass this argument would be rigorously true. For a running mass it is plausible that Z_ψ is still approximately one. This fact is true in all models of chiral symmetry breaking I know. A conservative estimate is

$$0.7 \le Z_\psi \le 1 \tag{6.39}$$

In the following we will set $Z_\psi = 1$ remembering that this not an exact statement. The results for all form factors have to be multiplied with Z_ψ.

6.6 The Axial Form Factors $G^{GI}_{1/2}(q)$

The form factor of the current $j_\Gamma = \bar{\psi}\Gamma\psi$ of a constituent quark can be reduced with the help of (6.38) to a 4 point function

$$tr_{CD}[T_{j_\Gamma}(p',p)\Gamma'] = \int d^4x\, d^4z\; e^{ip'x-ipz} tr_{CD}[\langle 0|\mathcal{T}\psi(x)\bar{\psi}(0)\Gamma\psi(0)\bar{\psi}(z)|0\rangle\Gamma'] = \tag{6.40}$$

$$= \int d\!\!{}^-{}^4 q \; \Pi_{\Gamma\Gamma'}(q-p, q-p', p, p')$$

The polarization functions $\Pi_{\Gamma\Gamma'}$ are calculated are defined and calculated in the instanton liquid model in [26] and other works. For $\Gamma = \gamma_\mu \gamma_5$ the connected part of the 4 point function is suppressed by $O(n_R^{1/2})$. In leading order in the instanton density only the disconnected part contributes and we get

$$T_{j^{GI}_{\mu 5}}(p', p) = S(p')\gamma_\mu \gamma_5 S(p) \tag{6.41}$$

Inserting (6.41) in (6.38) and comparison with (6.23) leads to

$$\langle p's'|J^{GI}_{\mu 5}(0)|ps\rangle = \bar{u}_{s'}(p')\gamma_\mu \gamma_5 u_s(p) \tag{6.42}$$

$$G_1^{GI}(q^2) = 1 \quad , \quad G_2^{GI}(q^2) = 0$$

Note, that $\Pi_{\Gamma\Gamma'}$ was calculated in singular gauge, but the connected part is suppressed in any gauge and the disconnected part only depends on the propagators, which cancel out anyway. The form factors $G^{GI}_{1/2}(q)$ are indeed gauge invariant. The result coincides with a model of free massive quarks. Further we see that the current is not conserved. Conservation demands $q^2 G_2 = M G_1$, which is clearly not satisfied by (6.42). In the one instanton approximation one can work from the very beginning with the effective 't Hooft vertex [17] which explicitly breaks the $U(1)$ symmetry and therefore contains the anomaly.

The result for the GI form factors (6.42), although not consistent with the experimental value, is *up to now* at least theoretical consistent.

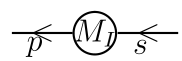
6.7 The Anomaly Form Factor $A(q)$ *

We will now calculate the anomaly form factor A. Using again the reduction formula with insertion of the anomaly current $B(0) = a(0)$ we have to calculate the 3 point function $T_a(p, s)$. In the instanton model the field operator $a(0)$ is replaced by a classical field $a_A(0)$ where $A = \sum_I A_I$ is a multi instanton configuration inserted in a. In a given background A the correlator can be written in the form

$$\langle 0|\mathcal{T}\psi(x)a(0)\bar{\psi}(z)|0\rangle_A = a_A(0)\langle 0|\mathcal{T}\psi(x)\bar{\psi}(z)|0\rangle_A = a_A(0)S_A(x,z) \tag{6.43}$$

where $S_A(x,z)$ is the quark propagator in the multi instanton background A. The r.h.s. has now to be averaged over the collective coordinates γ_I of all instantons. Without the factor $a_A(0)$ this is just the averaged quark propagator calculated in [26]. $a_A(y)$ is $2N_f$ times the topological charge density at spacetime point y. In the vicinity of an instanton of charge $Q_I = \pm 1$ the charge density has a positive/negative bump and is small elsewhere. Therefore $a_A(y)$ is only nonzero when there is at least one instanton near y. Let us fix exactly one instanton in the vicinity of $y = 0$. The orientation and charge of the remaining instantons can be averaged independently, but when averaging the locations z_I the domain near y has to be avoided. The next step is to assume 2 instantons near y and so on. The relative error we make by neglecting these further contributions and by forgetting about the restriction on z_I are both of $O(n_R)$. In leading order in the instanton density we can therefore fix one instanton near $y = 0$ and take only this contribution to $a_A(0)$ into account. The remaining instantons can be averaged as in the pure propagator case and the diagrams which have to be summed and averaged are the same except for the fixing of one instanton I. The propagator consists of a chain of instanton scatterings A_J $(J = 1 \dots N)$. Repeated scattering at this vertex is allowed. There are two cases: The first case is that all instantons left to all occurrences of instanton I are different to all instantons right to all occurrences of instanton I. In leading order in $1/N_c$ all instantons in the middle section from the first up to the last occurrence of A_I are different to the exterior instantons. The instantons on the left and on the right can be averaged independently leading to averaged multi-instanton propagators. Averaging the middle section, but fixing I leads to the effective vertex M_I. The free part of the correlator in momentum space is therefore

$$T_a^{free}(p,s) = \langle 2N_f Q_I(z_I)$$



$$\rangle_I =$$

$$= -2iN_f \hat{Q}(p-s)\sqrt{M_p M_s}\, S(p)\gamma_5 S(s) \qquad (6.44)$$

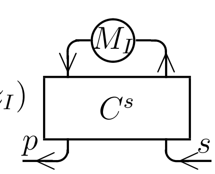
$$Q_I(z_I) = \frac{1}{2N_f} a_{A_I}(0) = \pm\frac{6}{\pi^2}\left(\frac{\rho}{z_I^2+\rho^2}\right)^4$$

$Q_I(z_I)$ is the charge density of one instanton of charge $Q_I = \pm 1$ and $\hat{Q}(q) = \frac{1}{2}(q\rho)^2 K_2(q\rho)$ its Fourier transform[3] for $Q_I = +1$. For $p^2 = s^2 = M^2$ the term $\sqrt{M_p M_s}$ is just the onshell mass M. Inserting (6.44) into (6.38) and comparison with (6.23) we get for the free part of the anomaly form factor:

$$A^{free}(q) = -N_f \hat{Q}(q) \quad , \quad A^{free}(0) = -N_f \qquad (6.45)$$

The second case is, that there are common instantons to the left and to the right of instanton I. The connected part of the correlator and the form factor are

$$T_a^{conn}(p,s) = \Big\langle 2N_f Q_I(z_I)$$



$$\Big\rangle_I = \qquad (6.46)$$

$$= -4(N_f-1)i\hat{Q}(p-s)C_5^s(p-s)F_5(p-s)\sqrt{M_p M_s}\, S(p)\gamma_5 S(s)$$

$$A^{conn}(q) = -2(N_f-1)\hat{Q}(q)C_5^s(q)F_5(q) \quad , \quad A^{conn}(0) = N_f - 1 \qquad (6.47)$$

For one flavor the connected part is zero as it should. For two flavors the result can easily derived by using the formulas of [26]. The total anomaly form factor for zero momentum transfer

$$A(0) = A^{free}(0) + A^{conn}(0) = -1 \qquad (6.48)$$

[3] I apologize for the overload of the symbol K: $K_2(q\rho)$ is a modified Bessel function, $K_\mu(x)$ is the anomaly current and $K_{1/2}(q)$ its form factors.

is independent of the number of flavors! This result is welcomed due to the following argument: The form factors of the axial singlet currents $j_{\mu 5}$ should not depend on any quark flavor which is not involved in the particle state. One expects that they are independent of N_f. Due to (6.22) matrix elements of $a(x)$ must then be independent of N_f too. But this is not obvious because $a(x)$ is explicitly proportional to N_f and the gluonic field is not flavor sensitive. The calculation given above shows how the quark interaction cancels the free part, which is proportional to N_f, so that the total form factor is independent of N_f at least at zero momentum transfer.

6.8 The Gluonic Form Factors $K^{GI}_{1/2}(q)$

Now we come to the calculation of $K_{1/2}(0)$. The previous calculation can be copied with minor changes. $a(0)$ has to be replaced by $K_\mu(0)$. This in turn induces the replacement

$$2Q(z_I) \rightsquigarrow G_\mu(z_I) := \frac{1}{N_f} K^\mu_{A_I}(0) \quad , \quad 2\hat{Q}(q) \rightsquigarrow \hat{G}_\mu(q) \tag{6.49}$$

$G_\mu(z_I)$ is $K_\mu(0)$ where the gauge field is an instanton centered at z_I of charge $Q_I = +1$ and $\hat{G}_\mu(q)$ is its Fourier transform. In regular gauge we get

$$\begin{aligned} G^{reg}_\mu(z) &= \frac{1}{N_f} K^\mu_{A_I^{reg}}(0) = -\frac{z_\mu(z^2+3\rho^2)}{\pi^2(z^2+\rho^2)^3} \\ \hat{G}^{reg}_\mu(q) &= -iq_\mu\rho^2 K_2(q\rho) \stackrel{q\to 0}{\longrightarrow} -2iq_\mu/q^2 \end{aligned} \tag{6.50}$$

With this replacement in (6.44) and (6.46) and comparison with (6.23) $K_{1/2}(0)$ can be extracted:

$$K^{reg}_1(q) = 0 \quad , \quad \lim_{q^2\to 0} \frac{q^2}{2M} K^{reg}_2(q) = 1 \tag{6.51}$$

In singular gauge we get

$$G^{sing}_\mu(z) = G^{reg}_\mu(z) + \frac{z_\mu}{\pi^2 z^4} \quad , \quad \hat{G}^{sing}_\mu(q) = \hat{G}^{reg}_\mu(q) + 2iq_\mu/q^2 \stackrel{q\to 0}{\longrightarrow} 0 \tag{6.52}$$

$$K^{sing}_1(q) = 0 \quad , \quad \lim_{q^2\to 0} \frac{q^2}{2M} K^{sing}_2(q) = 0$$

An apparent observation is, that the anomaly form factor $K_2(q)$ is gauge dependent and receives a massless pole in regular gauge. The reason for this is the gauge dependence of the anomaly current K_μ itself. One can show that the forward matrix elements $K_{1/2}(0)$ are GI for small gauge transformations. A gauge transformation is called small, when it can be smoothly deformed into the unit transformation. On the other hand the gauge transformation, which transforms an instanton from regular gauge to one in singular gauge is large, because the regular solution can not be smoothly deformed into a singular one due to the singularity.

The next striking observation is that the relation

$$K_1(q) - \frac{q}{2M} K_2(q) = A(q) \tag{6.53}$$

is violated in singular gauge as can be seen from (6.52)

$$K_1^{sing}(q) - \frac{q}{2M} K_2^{sing} \neq A(q) \tag{6.54}$$

Surface terms are the origin of this violation. For the derivation of (6.53) one has assumed the vanishing of surface terms. If one replaces the plane wave solution for the state by a wave packet, the state and therefore the matrix elements decrease sufficiently fast at spacial infinity and there are no surface terms. A experimental state is always a more or less localized wave packet rather than an exact plane wave. Therefore in regular gauge there are no surface terms and

$$K_1^{reg}(q) - \frac{q}{2M} K_2^{reg} = A(q) \tag{6.55}$$

is valid for all q. In order to work in singular gauge we have to choose a space-time manifold $I\!R^4 \backslash \{0\}$ to exclude the unphysical singularity. This small hole should not affect the physics at large distances. Therefore all coordinate space integrals are integrals over the domain $I\!R^4 \backslash B_\varepsilon(0)$. Partial integration can now lead to surface terms at zero. The surface term is non-zero in the case of G_μ^{sing} as can be seen from (6.52). This is the reason for the inequality (6.54). It is surprising that not the slowly decaying regular gauge field causes a surface term at infinity but the strong singularity at the instanton centers in singular gauge leads to surface terms and to a violation of (6.53).

The following conclusions should be drawn:

1. not to consider gauge dependent objects like $K_{1/2}(q)$ at all or
2. save the relation (6.53) by using regular gauge although this violates the philosophy of [23] or
3. modify relation (6.53) by including the surface terms and be careful when performing partial integrations.

In the following discussion we take position 2.

6.9 Discussion

Comparing the results for the form factors $\Delta\Sigma^{GI} = G_1^{GI}(0)$, $G_2^{GI}(0)$, $A(0)$, $K_1(0) = K_1^{reg}(0)$ and $K_2(0) = K_2^{reg}(0)$ summarized in the last row of table 6.4 we clearly see that they are in contradiction. It is not possible to determine the remaining form factors in a way that they are consistent with (6.24) and (6.34). The most obvious contradiction is $\Delta\Sigma^{GI} \neq A(0)$. An opposite sign of the anomaly would at least be theoretical consistent and would lead to the naive expectations. The only candidate for this violation of the axial ward identities is the neglect of the non-zeromodes. All other approximations respect the symmetries of QCD as discussed in Chapter 3.3.

Forte [53] has derived the relation $\Delta\Sigma + A(0) = 0$ in the instanton model in the case of one quark flavor in quenched approximation and density expansion. $\Delta\Sigma$ was identified with $\Delta\Sigma^c$ and $K_2(0)$ was assumed to be zero (although not explicitly stated). Therefore $A(0) = K_1(0)$ and from (6.32) one can arrive at the welcomed result $\Delta\Sigma^{GI} = 0$.

In section 6.8 I have shown that the anomaly contributes to K_2 and not to K_1. This is the first discrepancy. Further, in section 6.6 I have shown that $\Delta\Sigma$ has to be identified with $\Delta\Sigma^{GI}$. This is the second discrepancy. It may turn out that the inclusion of non-zeromodes removes the discrepancies in a way, that leads to a phenomenological welcomed small $\Delta\Sigma^{GI}$. In the one instanton approximation the inclusion is manageable and has been performed by [22] for the meson correlators. The consistent extension to the instanton liquid and to the quark form factors was not yet manageable.

These problems might be compared to calculations of the η' mass. A brute force method of calculating the axial singlet meson correlator and extracting $m_{\eta'}$ by a spectral fit is not successful too. More elaborate arguments, given in Section 6.6 allowed a successful determination of $m_{\eta'}$. The instanton model was only used as a motivation for a selfdual model of QCD. Maybe the same model is able to solve the proton spin problem rather than a brute force calculation.

It might also be possible that the spin problem can not be solved on the level of individual constituent quarks formed out of current quarks but is connected with a strong interaction in the axial singlet channel between different constituent quarks. This possibility in connection with instantons is discussed in [69].

Chapter 7

Gluon Mass

Up to now the instanton vacuum has been treated on the classical level. In this Chapter we compute the quantum fluctuations around the 1-instanton vacuum for the gluon propagator.

Section 7.1 gives a brief introduction to the problems and phenomenological consequences of massive gauge bosons. In the other Sections we will calculate the gluon propagator for different gauges. In background $\xi = 1$ gauge it has the simple form $S^{ab}_{\mu\nu} = \delta^{ab} g_{\mu\nu}/(p^2 - M(p)^2)$. In Section 7.2 we will extract the inverse propagator from the terms quadratic in the fluctuations around a background field. In Section 7.3 we will average this expression over relevant background fields and expand it for large momentum in a way to get the gluon mass $M(p)$. In Section 7.4 we explicitly calculate M in the multi-instanton background. For large momentum, however, this mass is cancelled out by terms which have been neglected. Furthermore, the mass is gauge dependent. To get reliable results for large as well as for small momentum we make in Section 7.5 a cluster expansion in the instanton density. For this purpose we need the gluon propagator in the 1-instanton background. The relevant formulas to construct this propagator are listed in Section 7.6. Because they are a bit lengthy we will restrict ourselves in Section 7.7 to the case of small momentum.

7.1 Introduction

The question, whether creation of a mass for gauge bosons is possible without gauge symmetry breaking, is very old. The explicit introduction of a gluon mass into the Lagrangian causes several difficulties. To preserve gauge invariance the gauge fields have to be coupled to massless scalar particles [30, 31], which decouple from physical matrix elements. The mass cannot be a constant, but must vanish for large momentum in order to ensure renormalizability of the theory (soft mass). Both phenomenons (massless scalar and soft gluon mass) occur in every model which dynamically creates a gluon mass. It should be mentioned that the terminology of a mass does not necessarily stand for a pole mass, but for the selfenergy. A dynamical mass means a non-vanishing selfenergy at $p^2 = 0$, i.e. if the massless pole has disappeared from the propagator. In [38] a gauge invariant selfenergy has been defined and computed. Solving the Schwinger Dyson equations leads to a mass of $500 \pm 200 MeV$. Due to asymptotic freedom the mass vanishes logarithmically for large momenta $M_{gluon}(p) \sim (\ln p^2)^{-12/11}$. Nonperturbative arguments lead to a decay $\frac{1}{p^2}(\ln p^2)^{12/11}$ [32, 33]. Finally it should be mentioned that a dynamical gluon mass is not in contradiction to confinement [34, 35, 36, 37].

Apart from regularizing all infrared divergences the gluon mass has a series of phenomenological consequences. A direct consequence are gluon balls. In the simplest picture in which glueballs consist of N independent constituent quarks the mass is of the order of $N \cdot M_{gluon}$. Hadrons with gluons can also be constructed ($\bar{q}qg$, $qqqg$, $\ldots$). Of course, the gluons are highly virtual due to the strong interaction. It is not even known whether this picture is qualitatively correct. One can only refer to the quark sector and the success of the constituent quark model. Furthermore, a gluon mass modifies the transverse momentum distribution of gluon jets. With a theoretically computed gluon mass one has a natural energy cutoff. A gluon mass is also welcomed for instanton physics: In a massive gauge theory instantons of a size $\rho \geq M_{gluon}^{-1}$ are exponentially suppressed. The infrared problem disappears. Under the assumption of the Instanton Liquid Model with independent instantons of radius $\rho = 600 MeV^{-1}$ and density $n = (200 MeV)^4$ one gets a gluon mass of $M_{gluon} = 480$ MeV near the cutoff scale $\rho^{-1} = 600 MeV$. Note that this possible solution of the infrared problem is different from the standard hope of finding a repulsive interaction between instantons [24, 25, 20].

For a quantitative theory one has to generalize the calculation of unphysical gluon propagators to gauge invariant matrix elements, like glueball correlators. This is a problem in all works up to now (including this one). The methods to achieve a gluon mass are too complicated to be applied to physical quantities. For this reason it is difficult to make the above mentioned physical applications more quantitative.

In the following Sections we will compute the averaged quantum gluon propagator for various gauges in the multi-instanton background. Although explicit expressions are known for a long time [18] they have not been used in phenomenological applications since they are rather complex. Till now the instanton background has been treated at the classical level. Even the 1-instanton action [17] is of relatively low practical importance due to the infrared problem. On the other hand quantum corrections to gluonic n point functions can be large.

7.2 Gluon Propagator *

The first task to obtain a formal expression for the gluon propagator in a background field is to expand $\mathcal{L}_{QCD}[\bar{A}+B]$ in the fluctuations B_μ^a around our background $\bar{A}_\mu^a$. The term quadratic in B_μ^a is then by definition the inverse gluon propagator. For $\bar{A}_\mu^a$ we will later use our instanton gas. We will use the QCD-Lagrangian

$$\mathcal{L}_{QCD} = \frac{1}{4g^2} G_{\mu\nu}^a G_a^{\mu\nu} \quad , \quad G_{\mu\nu}^a(A) = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{abc} A_\mu^b A_\nu^c$$

where we have rescaled the fields in such a way that the coupling-constant-dependence is in front of $\mathcal{L}$. We will entirely work in Euclidian space with metric $\delta_{\mu\nu}$ instead of $g_{\mu\nu}$ because instantons do not make sense in Minkowski space. At the very end we can simply rotate back to Minkowski space. With these conventions

$$g^2\mathcal{L}_{QCD}(\bar{A}+B) = \frac{1}{4}G^a_{\mu\nu}(\bar{A}+B)G^{\mu\nu}_a(\bar{A}+B) =$$

$$= \frac{1}{4}\overbrace{\bar{G}^a_{\mu\nu}\bar{G}^{\mu\nu}_a}^{O(B^0)} + \overbrace{B^a_\mu\bar{D}^{ab}_\nu\bar{G}^{\mu\nu}_a}^{O(B^1)} + \frac{1}{2}\overbrace{B^a_\mu(-\bar{D}^{ac}_\rho\bar{D}^{cb}_\rho\delta_{\mu\nu} - 2f_{acb}\bar{G}^c_{\mu\nu} + \bar{D}^{ac}_\mu\bar{D}^{cb}_\nu)B^b_\nu}^{O(B^2)} +$$

$$+ \underbrace{f_{abc}B^b_\mu B^c_\nu\bar{D}^{ad}_\mu B^d_\nu}_{O(B^3)} + \frac{1}{4}\underbrace{f_{abc}B^b_\mu B^c_\nu f_{ade}B^d_\mu B^e_\nu}_{O(B^4)} + \partial_\mu(\ldots) \quad , \tag{7.1}$$

where $\bar{D}^{ab}_\mu = \partial_\mu\delta_{ab} + f_{acb}\bar{A}^c_\mu$ is the covariant derivative with $\bar{A}$ inserted instead of A; similarly $\bar{G}^a_{\mu\nu} = G^a_{\mu\nu}(\bar{A})$. As usual the terms which are total derivatives disappear after integrating $\mathcal{L}$. We see that if $\bar{A}$ solves the equation of motion the linear term vanishes, as it should be. If we choose background gauge $\bar{D}^{ac}_\nu B^c_\nu = 0$ the last term of the $O(B^2)$-contribution vanishes. Now we can read the inverse gluon propagator from the terms quadratic in B (from now on we will omit the bars over A, G and D because the unbared objects won't be needed further):

$$(S^{-1})^{ab}_{\mu\nu} = \frac{1}{g^2}(-D^{ac}_\rho D^{cb}_\rho\delta_{\mu\nu} - 2f_{acb}G^c_{\mu\nu}) \quad . \tag{7.2}$$

We will also omit the $1/g^2$ in front of the propagator which is a result of the rescaling of fields anyway. For further manipulations some abbreviations are useful:

$$G_{\mu\nu} = F^cG^c_{\mu\nu} \quad , \quad A_\mu = F^cA^c_\mu \quad ,$$

$$(F^c)_{ab} = if_{acb} \quad , \quad [F^a,F^b] = if_{abc}F^c \quad , \quad tr_cF^aF^b = N_c\delta^{ab} \quad , \quad N_c = 3 \quad ,$$

$$(\hat{P}_\mu)^{ab} = iD^{ab}_\mu \quad , \quad \hat{p}_\mu = i\partial_\mu \quad , \quad \hat{P}_\mu = \hat{p}_\mu + A_\mu \quad ,$$

$$\hat{p}_\mu X = [\hat{p}_\mu, X] + X\hat{p}_\mu = i(\partial_\mu X) + X\hat{p}_\mu \quad . \tag{7.3}$$

The last equation has only been quoted to show that $\hat{p}$ and $\hat{P}$ will be used in operator sense. F^a are the generators in adjoint representation and f_{abc} are the structure constants of the color gauge group $SU(N_c)$. With these abbreviations we can now write

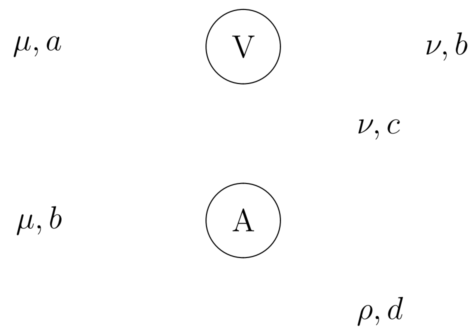
$$S^{-1}_{\mu\nu} = \hat{P}^2\delta_{\mu\nu} + 2iG_{\mu\nu} = (\hat{p}^2 + \hat{p}\cdot A + A\cdot\hat{p} + A^2)\delta_{\mu\nu} + 2iG_{\mu\nu} = (S^{-1}_0 + V)_{\mu\nu} \quad ,$$

$$S^0_{\mu\nu} = \delta_{\mu\nu}/\hat{p}^2 \quad , \quad V_{\mu\nu} = (A^2 + \hat{p}\cdot A + A\cdot\hat{p})\delta_{\mu\nu} + 2iG_{\mu\nu} \quad . \tag{7.4}$$

S_0 is the free gluon propagator with no background and V can be interpreted as an interaction potential caused by the background. The QCD-Lagrangian in the background 7.1 can be written in another form more suitable for ordinary perturbation theory:

$$g^2\mathcal{L}_{QCD} = \frac{1}{4}G^a_{\mu\nu}(B)G^{\mu\nu}_a(B) + \frac{1}{2}B^a_\mu V^{ab}_{\mu\nu}B^b_\nu + f_{abc}f_{aed}B^b_\mu B^c_\nu B^d_v\bar{A}^\mu_e \tag{7.5}$$

The terms independent and linear in B have been omitted. Constant terms are irrelevant for the dynamics and terms linear in B are zero if A solves the QCD equations of motion. The new terms due to the background create two additional Feynman diagrams



$$= V^{ab}_{\mu\nu}$$

$$= g f_{abc} f_{aed} A^e_\mu \delta_{\nu\rho}$$

It should be noted that for small coupling constant g the second graph can be treated perturbatively but not the first. So our main concentration lies on the first term.

7.3 Propagator in Statistical Background *

The next step is to use some approximation scheme to calculate the propagator

$$S = (S_0^{-1} + V)^{-1} = S_0(\mathbb{1} + T)^{-1} \quad , \quad T = V S_0 \quad . \tag{7.6}$$

For large momentum p, S_0 and therefore T are small and we can expand S in powers of T:

$$S = S_0(\mathbb{1} - T + T^2 - T^3 + \ldots) \quad . \tag{7.7}$$

Note that $S(x,y) = \langle x|S|y\rangle$ is generally not invariant under translations and rotations because the background A_μ and therefore T are not. Actually we are not interested in the propagator for a particular background configuration, but only in the average over all relevant configurations. Here we do not mean the functional integration over quantum fluctuations around the empty vacuum, but fields other than the perturbative $A_\mu = 0$-vacuum which minimize the action $\int \mathcal{L}\, dx$.

$$\overline{S} = S_0(\mathbb{1} - \overline{T} + \overline{T^2} - \overline{T^3} + \ldots) \quad , \tag{7.8}$$

where the bar denotes averaging over relevant configurations. Details are specified below. If the background is statistically invariant under translations then $\overline{S}$ is translationally invariant and therefore diagonal in momentum space $\overline{S}(p,q) = \langle p|\overline{S}|q\rangle = \overline{S}(p)\bar{\delta}(p-q)$. If we would evaluate the terms in the series we would get an expansion of $\overline{S}$ in the form

$$\overline{S}(p) = 1/p^2 + c_1/p^4 + c_2/p^6 + \ldots \tag{7.9}$$

and the pole remains at zero — and gets even worse with higher terms. What we want is $\overline{S}$ in a form like $\overline{S}(p) = (p^2 + M(p)^2)^{-1}$ with $M(p)$ bounded for large and small p and interpreted as momentum dependent gluon mass[1]. So let's invert (7.8)

$$\overline{S}^{-1} = \hat{p}^2 + M(\hat{p})^2 = (\mathbb{1} - \overline{T} + \overline{T^2} - \overline{T^3} + \ldots)^{-1} S_0^{-1} \tag{7.10}$$

[1]The $''+''$ in front of M will change to the more familiar $''-''$ when we rotate back from Euclidian space to Minkowski space.

and expand it once again in T. Without averaging this would just be a geometrical series which, expanded, would give the original formula $S^{-1} = S_0^{-1} + V$. With averaging the different terms are now in no relation and expanding and sorting with respect to powers of T yields

$$\begin{aligned}\overline{S}^{-1} &= (\mathbb{1} + \overline{T} - (\overline{T^2} - \overline{T}^2) + (\overline{T^3} - \overline{T}\,\overline{T^2} - \overline{T^2}\,\overline{T} + \overline{T}^3))S_0^{-1} + O(T^4) \qquad (7.11)\\ &= S_0^{-1} + \overline{V} - (\overline{VS_0V} - \overline{V}S_0\overline{V}) + \dots\\ &= S_0^{-1} + M^2, \quad M^2 = M_1^2 - M_2^2 + \dots, \quad M_1^2 = \overline{V}, \quad M_2^2 = \overline{VS_0V} - \overline{V}S_0\overline{V}.\end{aligned}$$

In the next section we introduce the concepts of instanton gas calculation to determine M_1, which is actually very simple.

7.4 A Naive Estimate of the Gluon Mass *

In the Instanton Liquid Model, in which $A = \sum_I A_I$ is a sum of instantons in singular gauge with fixed radius $\rho = 600\mathrm{MeV}^{-1}$, the scatter amplitude

$$\overline{V}_{\mu\nu} = (\overline{A^2} + \hat{p}\cdot\overline{A} + \overline{A}\cdot\hat{p})\delta_{\mu\nu} + 2i\overline{G}_{\mu\nu} \ :$$

can be easily computed. A short calculation shows [40]

$$\overline{A_\mu} = \langle A_{I_\mu}\rangle_I = 0 \quad , \quad \overline{G_{\mu\nu}} = 0 \qquad (7.12)$$

For $M_1^2 = \overline{V} = \overline{A^2}$ we get

$$(\overline{A^2}) = \frac{12\pi^2 N_c}{N_c^2 - 1} n\rho^2 \delta_{ab} \qquad (7.13)$$

For $M_1^2 = \overline{V} = \overline{A^2}$ this leads to

$$M_1 = \sqrt{\frac{12\pi^2 N_c}{N_c^2 - 1}}\sqrt{n}\rho \approx 420\,\mathrm{MeV} \quad , \qquad (7.14)$$

where we used the instanton density $n = (200\mathrm{MeV})^4$. This value coincides very well with the results of [38] und [39].

For large momentum there is a term M_2 which completely cancels M_1. A vanishing mass for large momentum is expected due to asymptotic freedom. To see this in detail one can count the number of $\hat{p}'s$ occurring in M_2^2 which will give us the dominant behaviour of M_2^2 for large p. $\overline{V}S_0\overline{V} = M_1^4/p^2$ vanishes for large p but in principle V contains a $\hat{p}$ in the numerator $(\hat{p}A)$ and $S_0 = 1/\hat{p}^2$ and $\overline{VS_0V}$ could be finite for large p. To be more definite consider

$$VS_0V = (pA + Ap)S_0(pA + Ap) + \text{other terms} = 4A_\mu \frac{p_\mu p_\nu}{p^2} A_\nu + \dots$$

where we have used $[p, A_I] = i\partial_\mu A_I^\mu = 0$.

$$\langle x|VS_0V|y\rangle = 4A_\mu(x)\langle x|\frac{p_\mu p_\nu}{p^2}|y\rangle A_\nu(y) + \ldots$$

From $\delta_{\mu\nu}\langle x|p_\mu p_\nu/p^2|y\rangle = \langle x|y\rangle = \delta(x-y)$we can conclude that$\langle x|p_\mu p_\nu/p^2|y\rangle = \frac{1}{4}\delta_{\mu\nu}\delta(x-y)$ + traceless terms.

$$\langle x|VS_0V|y\rangle = A_\mu(x)A_\mu(x)\delta(x-y) + \ldots = \langle x|A^2|y\rangle + \ldots$$

So $M_2^2 = \overline{A^2} + \ldots = M_1^2 + \ldots$ contains a term which fully cancels M_1^2 calculated above. Chapter 8 explains why such cancellations are expected in singular gauge.

In the next section we present a systematic expansion of the propagator in the instanton density.

7.5 Expansion in the Instanton Density *

In this section we will make an expansion of the gluon propagator in the instanton density n which will be valid for all Euclidian momenta p especially for small p. Furthermore, the result will be Gauge invariant.

The average of an arbitrary power of the 1-instanton field is proportional to the inverse volume $\overline{A_I^n} \sim \frac{1}{V}$ for $n \geq 1$. So, for example, the expansion of the square of the multi-instanton configuration in the instanton density is

$$\overline{A^2} = \sum_{I=1}^{N}\overline{A^2} + \underbrace{N\overline{A_I}}_{O(n)}\underbrace{(N-1)\overline{A_I}}_{O(n)} = \underbrace{N\overline{A_I^2}}_{O(n)} + O(n^2) \tag{7.15}$$

and more general

$$\overline{A^n} = N\overline{A_I^n} + O(n^2) \quad , \quad \overline{V^n} = N\overline{V_I^n} + O(n^2) \quad \text{for} \quad n \geq 1, \tag{7.16}$$

where $V_I = V(A_I)$ defined in (7.4) with A replaced by A_I and similar for T. Let us now sum up all terms in (7.11) linear in n:

$$\begin{aligned}\overline{S}^{-1} &= (\mathbb{1} + \overline{T} - \overline{T^2} + \overline{T^3} - \ldots)S_0^{-1} + O(n^2) \\ &= (\mathbb{1} + N(\overline{T_I} - \overline{T_I^2} + \overline{T_I^3} - \ldots))S_0^{-1} + O(n^2) \\ &= (\mathbb{1} + N\overline{T_{eff}})S_0^{-1} + O(n^2) = S_0^{-1} + N\overline{V_{eff}} + O(n^2) \end{aligned} \tag{7.17}$$

$$\begin{aligned}T_{eff} &= T_I - T_I^2 + T_I^3 - \ldots = T_I - T_I(T_I - T_I^2 + \ldots) = T_I - T_IT_{eff} \\ V_{eff} &= V_I - V_IS_0V_{eff} \quad \Longrightarrow \quad V_{eff} = S_0^{-1}(S_0 - S_I)S_0^{-1} \quad \text{with} \qquad (7.18) \\ S_I^{-1} &= S_0^{-1} + V_I \quad \text{is the propagator in the 1-instanton background.} \qquad (7.19)\end{aligned}$$

More generally we can make a cluster expansion of an arbitrary function of A in the following form

$$\overline{f(A_1+\ldots+A_N)} = \overline{f(A)} = \sum_{l=0}^{N} \binom{N}{l} \overline{\sum_{k=0..l} (-)^{l-k} \binom{l}{k} f(A_1+\ldots+A_k)} = \tag{7.20}$$

$$= \underbrace{f(0)}_{O(1)} + \underbrace{N\overline{f(A_1)-f(0)}}_{O(n)} + \frac{1}{2}\underbrace{N(N-1)\overline{f(A_1+A_2)-2f(A_2)+f(0)}}_{O(n^2)} + O(n^3)$$

where the first line is an identity even without averaging. Inserting a Taylor expansion for f in the second line and using the indistinguishability of different instantons one can see that all monomials in the k-th term contain more or equal than k different instantons. So the average will factorize in k factors each proportional to n and therefore the k-th term is indeed proportional to n^k. It is easy to generalize (7.20) for two or more species of fields. If A is a field of N_I instantons *and* $N_{\bar{I}}$ anti-instantons we get to first order in n

$$\overline{f(A)} = f(0) + N_I\overline{f(A_I)-f(0)} + N_{\bar{I}}\overline{f(A_{\bar{I}})-f(0)} + O(n^2). \tag{7.21}$$

If we insert the propagator S in (7.21) we get

$$\overline{S} = \overline{S(A)} = S(0) + N\overline{S(A_I)-S(0)} + O(n^2) = S_0 + N\overline{S_I - S_0} + O(n^2) \tag{7.22}$$

which is after inversion up to $O(n^2)$ just (7.17).

7.6 QCD Propagators *

In Section 5 we have seen that it is enough to know the 1-instanton propagator to calculate $\overline{S}$ to first order in n. Luckily this propagator is known, unfortunately it is a bit lengthy and suffers from a divergence.

The gluon propagator $S^{ab}_{I\mu\nu}$ with spin $S=1$ in adjoint color[2] representation ($C=1$) can be constructed out of the ghost propagator Δ^{ab}_I ($S=0, C=1$) which is explicitly known in the 1-instanton background. The general formulas how to construct a propagator of given spin S out of the corresponding scalar propagator with same color C in a selfdual background are shown below. They are derived and more thoroughly discussed in [18].

Notations:

$$A_\mu = T^a A^a_\mu \quad , \quad [T^a,T^b] = i\epsilon_{abc}T^c \quad , \quad T^a = \text{generator of} \quad SU(2)_c$$

$$T^a = \left\{ \begin{array}{cll} 0 & \text{in scalar} & (C=0) \\ \tau^a/2 & \text{in fundamental} & (C=\frac{1}{2}) \\ i\epsilon_{\cdot a \cdot} & \text{in adjoint} & (C=1) \end{array} \right\} \text{representation} \tag{7.23}$$

$$P_\mu = p_\mu + A_\mu \quad , \quad p_\mu = i\partial_\mu \quad , \quad \tilde{G}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G_{\rho\sigma} \quad , \quad \{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu}$$

[2] Often called isospin [18]

Spin 0 *propagator* $\tilde{\Delta}$*:*

$$\tilde{\Delta}^{-1} = P^2 \quad \Longrightarrow \quad \tilde{\Delta} = P^{-2} \tag{7.24}$$

Spin $\frac{1}{2}$ *propagator* S*: (not used but only stated for completeness)*

$$S^{-1} = \not{P} = \gamma_\mu P^\mu \quad \Longrightarrow \quad S = \not{P}\tilde{\Delta}\frac{1+\gamma_5}{2} + \tilde{\Delta}\not{P}\frac{1-\gamma_5}{2} \quad \text{for} \quad G_{\mu\nu} = \tilde{G}_{\mu\nu} \tag{7.25}$$

Spin 1 *Propagator* S*:*

$$S^{-1}_{\mu\nu} = P^2\tilde{\Delta}_{\mu\nu} + 2iG_{\mu\nu} - (1-\frac{1}{\xi})P_\mu P_\nu \quad \Longrightarrow$$

$$S_{\mu\nu} = q_{\mu\nu\rho\sigma}P_\rho\tilde{\Delta}^2 P_\sigma - (1-\xi)P_\mu\tilde{\Delta}^2 P_\nu \quad \text{for} \quad G_{\mu\nu} = \tilde{G}_{\mu\nu}, \tag{7.26}$$

$$q_{\mu\nu\rho\sigma} = \delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} + \epsilon_{\mu\nu\rho\sigma} \quad .$$

There are some comments in order. The above formulas are valid for an arbitrary color representation, we will need them only for $C = 1$. For $S \neq 1$ there are zeromodes and the propagator is only the inverse of the kernel in a subspace orthogonal to the zeromodes. The implications and problems will be discussed in Section 8 when they show up explicitly. The Spin 1 kernel is the quadratic term of the QCD-Langrangian (7.1) with gauge fixing term $\frac{1}{2\xi}(D^{ab}_\mu B^b_\mu)^2$ in slight generalization to the $\xi = 1$ case considered previously.

With

$$\Pi(x) = 1 + \frac{\rho^2}{x^2} \quad , \quad F(x,y) = 1 + \rho^2\frac{(\tau x)}{x^2}\frac{(\tau^\dagger y)}{y^2} \quad , \tag{7.27}$$

$$\tau_\mu = (\vec{\tau}, i) \quad , \quad \tau^\dagger_\mu = (\vec{\tau}, -i) \quad , \quad \tau_\mu\tau^\dagger_\nu = \delta_{\mu\nu} + i\bar{\eta}_{a\mu\nu}\tau_a$$

we can write the ghost propagator for 1 instanton in the form [18]

$$\begin{aligned}
\Delta^{ab}_I(x,y) &= \frac{\frac{1}{2}\mathrm{tr}\ \tau_a F(x,y)\tau_b F(y,x)}{4\pi^2(x-y)^2\Pi(x)\Pi(y)} = \\
&= \frac{\delta_{ab}}{4\pi^2(x-y)^2} - \frac{\rho^2\delta_{ab}}{4\pi^2(x^2+\rho^2)(y^2+\rho^2)} + \frac{2\rho^2\epsilon_{abc}\bar{\eta}_{c\mu\nu}x_\mu y_\nu}{4\pi^2(x-y)^2(x^2+\rho^2)(y^2+\rho^2)} \\
&\quad + \frac{2\rho^4(((xy)^2 - x^2y^2)\delta_{ab} + \epsilon_{abc}\bar{\eta}_{c\mu\nu}x_\mu y_\nu + \bar{\eta}_{a\mu\nu}\bar{\eta}_{b\rho\sigma}x_\mu y_\nu x_\rho y_\sigma}{4\pi^2(x-y)^2(x^2+\rho^2)x^2(y^2+\rho^2)y^2}
\end{aligned} \tag{7.28}$$

For simplicity we have placed the instanton at the origin $z_I = 0$ in standard orientation $O^{ab} = \delta^{ab}$. It is possible to write down the gluon propagator $S^{ab}_{I\mu\nu}$ explicitly in which we are finally interested in using (7.26) but the expression will be rather lengthy. While Δ^{ab}_I can be averaged and represented in momentum space exactly with the help of modified Bessel functions this seems not to be possible for $S^{ab}_{I\mu\nu}$ and we have to expand $\overline{S}$ for large and/or small momenta (or work much harder to solve the complicated integrals in terms of special functions). So we will never use the full expression for S.

7.7 Propagators for Small Momentum *

The calculation simplifies significantly if we expand $\overline{\Delta}(p)$ or $\overline{S}(p)$ for small momentum. Because p always occurs in the dimensionless quantity $(p\rho)$ the lowest order in p can be found by keeping only terms of lowest order in ρ or differently stated: Small p corresponds to large x and for $x \gg \rho$ ρ is negligible. To warm up let us start with Δ from (7.28):

$$\Delta_I^{ab} = \Delta_0^{ab} - \rho^2 W^{ab} + O(\rho^4) \tag{7.29}$$

$$\Delta_0^{ab}(x,y) = \frac{\delta_{ab}}{4\pi^2(x-y)^2}$$

$$W^{ab}(x,y) = \frac{\delta_{ab}}{4\pi^2 x^2 y^2} + \frac{2\epsilon_{abc}\bar{\eta}_{c\mu\nu}x_\mu y_\nu}{4\pi^2(x-y)^2 x^2 y^2}$$

$\Delta_0(p) = 1/p^2$ is the free ($A_\mu^a \equiv 0$) ghost propagator. After reintroducing the instanton position z we can now average $\Delta_0 - \Delta_I$. The ϵ-term in W will be killed by averaging over the instanton orientation:

$$\langle x|\overline{\Delta_0^{ab} - \Delta_I^{ab}}|y\rangle = 4\pi^2\delta_{ab}\rho^2\frac{1}{V_4}\int\frac{d^4z}{4\pi^2(x-z)^2 4\pi^2(z-y)^2} \quad , \tag{7.30}$$

The last integral is infrared divergent but noticing that $1/4\pi^2(x-y)^2$ is the Fourier transformation of $1/p^2$ we can rewrite the above expression in the form

$$V_4\langle x|\overline{\Delta_0 - \Delta_I}|y\rangle = 4\pi^2\rho^2\int d^4z\langle x|\frac{1}{p^2}|z\rangle\langle z|\frac{1}{p^2}|y\rangle = 4\pi^2\rho^2\langle x|\frac{1}{p^4}|y\rangle$$

The divergence is now hidden in the fact that the coordinate representation of p^{-4} does not exist. In fact we are not interested in the coordinate representation but in the momentum representation and so the divergence is spurious. Inserting Δ in (7.18) we get

$$\overline{V}_{eff}^{ghost} = \Delta_0^{-1}\ \overline{\Delta_0 - \Delta_I}\ \Delta_0^{-1} = p^2\frac{4\pi^2\rho^2}{V_4}\frac{1}{p^4}p^2 = \frac{1}{V_4}4\pi^2\rho^2 = \rho^2 p^4\overline{W} \tag{7.31}$$

and lead in the $SU(2)_c$ case to a ghost mass of

$$M_{ghost}^2(p=0) = N\overline{V}_{eff} = 4\pi^2\rho^2 n$$

which has to be inserted into the ghost propagator $\overline{\Delta}_{ab} = \delta_{ab}(p^2 + M_{ghost}^2(p))^{-1}$. Generalization to the case of N_c colors gives

$$M_{ghost}(p=0) = 2\pi\rho\sqrt{2n_R} = 8.9\rho\sqrt{n_R} = 340\,\text{MeV} \tag{7.32}$$

where $n_R := n/N_c$ is of the order of $O(N_c^0)$. The most important insight is that $M \neq 0$. A scalar particle in adjunct color representation achieves a dynamical mass in the instanton background.

Let us now calculate the gluon mass at zero momentum along the same lines. To do this we must insert (7.29) into (7.26). Expanding also P_μ up to order ρ^2

$$P_\mu = p_\mu + \rho^2 \breve{A}_\mu + O(\rho^4) \quad , \quad \breve{A}_\mu = \frac{2F_a \bar{\eta}_{a\mu\nu} x_\nu}{x^4}$$

we get

$$P_\mu \Delta_I^2 P_\nu = p_\mu \Delta_0 p_\nu - \rho^2 p_\mu (\Delta_0 W + W\Delta_0) p_\nu + \rho^2 (p_\mu \Delta_0 \breve{A}_\nu + \breve{A}_\mu \Delta_0 p_\nu) + O(\rho^4)$$

The free gluon propagator in R_ξ-gauge is well known or can be obtained from (7.26) setting $G_{\mu\nu} = 0$:

$$S^0_{\mu\nu} = q_{\mu\nu\rho\sigma} p_\rho \Delta_0^2 p_\sigma - (1-\xi) p_\mu \Delta_0^2 p_\nu = \frac{1}{p^2}(\delta_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2})$$

$$S^0_{\mu\nu} - S^I_{\mu\nu} = \rho^2 [q_{\mu\nu\rho\sigma} p_\rho (\Delta_0 W + W\Delta_0) p_\sigma - (1-\xi) p_\mu (\Delta_0 W + W \Delta_0) p_\mu]$$

$$\overline{S^0_{\mu\nu} - S^I_{\mu\nu}} = 2\rho^2 \overline{W} (p^2 \delta_{\mu\nu} - (1-\xi) p_\mu p_\nu) = 2\rho^2 p^2 \overline{W} S^0_{\mu\nu}$$

where we have used in the last line that p commutes with averaged quantities like $\overline{W}$.

$$\overline{V}^{gluon}_{eff} = S_0^{-1} \overline{S_0 - S_I} S_0^{-1} = 2\rho^2 p^2 S_0^{-1} \overline{W}$$

$$\overline{S}^{-1} = S_0^{-1} + N\overline{V^{gluon}_{eff}} = S_0^{-1}(1 + 2M^2_{ghost}(0)/p^2)$$

$$\overline{S} = \frac{p^2 S_0}{p^2 + M^2_{ghost}(0)} = \frac{\delta_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}}{p^2 + M^2_{gluon}(p^2)} \quad \text{with}$$

$$M^2_{gluon}(p^2) = 2M^2_{ghost}(0) + O(p^2)$$

We got the interesting result that for momentum transfer 0 the gluon mass is a factor of $\sqrt{2}$ larger than the ghost mass. Maybe the relation $M_{gluon} \approx 2M_{ghost}$ is valid even for larger p. The gluon mass for momentum transfer zero is

$$M_{gluon}(p=0) = 4\pi\rho\sqrt{n_R} = 12.6\rho\sqrt{n_R} = 480\,\text{MeV} \tag{7.33}$$

7.8 Zeromodes *

In our whole calculation we have ignored the zeromodes. All gluon field fluctuations must be orthogonal to these which is achieved by using a gluon propagator orthogonal to the zeromodes or otherwise stated by subtracting out from (7.26) the projections on the zeromode subspace. This procedure also deletes a divergence coming from the second term in (7.28) squared. There are some further problems to be solved because the scalar product of the zeromodes with the propagator does not exist [18]. But all this concerns a term which is proportional to ρ^4 which was irrelevant in our zero momentum approximation. Also the principle mechanism of mass generation does not depend on the zeromodes which can be seen from the ghost propagator because in the spin 0 case there are no zeromodes. This should be contrasted with the fermionic case where the zeromode are the main ingredient for mass generation and the so called zeromode approximation simplifies calculations enormously.

7.9 Conclusions and further developments

We have computed the gluon propagator in the Instanton Liquid Model for small momentum p in leading order in the instanton density. A mass independent of N_c has been extracted. Furthermore, the mass is gauge independent, at least for $o^2 = 0$ within the class of R_ξ gauges. As expected, for $\xi \to 0$, the gluon propagator is transversal.

A massless pole remains for $\xi \neq 1$ due to the $(1-p)p_\mu p_\nu/p^2$ term. We expect that the massless state, which is associated with this pole, decouples from physical matrix elements, as in conventional perturbation theory.

The result is in agreement with the values calculated by Cornwall ($500 \pm 200 MeV$) [38] or extracted from pp scattering ($370 MeV$) [39].

The next step may be the calculation of some gauge invariant correlation function like a glueball correlator or topological susceptibility. As usual this involves products and integrals over propagators but with the difference that we have to use (7.26) instead of the simple free propagator. Without further simplification this may cause a headache.

Chapter 8

Gauge Invariant Quark Propagator

A variety of predictions concerning chiral symmetry breaking and concerning the lightest hadrons in various channels can be made within the instanton liquid model. Although there are various attempts to derive this model from first principles, it is still an open question, whether instantons melt or not. Thus the infrared problem remains unsolved. In this chapter I attempt to make a few predictions which do not rely on the Instanton Liquid Model, but exploit the theoretical one-loop density $D(\rho)$.

The quark condensate will be calculated in Section 8.5 and 8.6. I get a finite result by choosing an appropriate gauge and performing a self energy resummation without introducing an infrared cutoff in the instanton density.

Because the finiteness essentially depends on the choice of gauge, I give a more general discussion in Section 8.1, 8.2 and 8.3 of how to choose a gauge when calculating gauge dependent quantities. The quark propagator in the background of one instanton in the well known regular and singular gauge is compared to a gauge invariant propagator, for which explicit expressions are calculated in Section 8.1.

8.1 Generalities On the Choice of Gauge

Gauge symmetry is a rather large symmetry, an infinite product of $SU(N_c)$ in the case of QCD. A physicist is always happy of having symmetries because they can be exploited to make predictions even without solving the theory. Gauge symmetry is necessary to get a physical vector particle spectrum. As long as one does not make an approximation which manifestly breaks gauge symmetry one can choose a comfortable gauge for calculations because the result is GI. But it is very difficult *not* to break GI, especially in a non-abelian gauge theory. It is not easy to find a GI regularization and furthermore, the gluon propagator, the primary object in perturbation theory, is not GI. Of course it is meanwhile well known how to perform GI calculations in every order perturbation theory using FP-ghosts and dimensional regularization. Every new approach beyond perturbation theory is again confronted with the problem of GI. In lattice theory the Wilson action had to be invented. In Schwinger-Dyson and Bethe-Salpeter type selfconsistency equations GI is still an open problem. In instanton physics when going beyond the one instanton approximation the choice of gauge is also important. This will be discussed in the next paragraph. There is a related problem when considering non-GI objects from the very beginning like the gluon or quark propagator. Strictly speaking they are only *defined* when relying to a certain gauge. In principle one should not give them any physical meaning at all. Often one is tempted to do so and therefore it is necessary to give some motivation of choosing this or that gauge.

8.2 A Natural Gauge

The gauge field A_μ^a describes the connection between neighboring vector bundles over the spacetime manifold $\mathbb{R}^4$. Thus a choice of gauge is like the choice of a coordinate system in general relativity with connection $\Gamma^\mu{}_{\nu\rho}$. When choosing a crooked coordinate system, although being in a smooth universe, there will appear fictitious accelerations towering above the real physical accelerations

$$\ddot{x}^\mu_{phys} = \ddot{x}^\mu_{fict} + \Gamma^\mu{}_{\nu\rho}\dot{x}^\nu\dot{x}^\rho \quad .$$

When making general covariant calculations these fictitious accelerations and the Γ contribution will cancel out thus leading to the correct small result. But the slightest unsystematic approximation will produce gross errors. The natural solution of this problem is to use a coordinate system as smooth as possible to avoid fictitious accelerations, e.g. to choose $\Gamma^\mu{}_{\nu\rho}$ as small as possible. To make this statement more quantitative we may try to minimize $(\ddot{x}^\mu_{phys} - \ddot{x}^\mu_{fict})^2$ simultaneously for all curves. This is done by choosing a coordinate system which minimizes[1]

$$||\Gamma||^2 \ := \int \Gamma_\mu{}^{\nu\rho}\Gamma^\mu{}_{\nu\rho}\, d^4x$$

This obviously measures the crookedness of the coordinate system.

Let us now transfer this to QCD. The analog norm for the gauge potential is

$$||A||^2 \ := \int tr_c A_\mu A^\mu \, d^4x$$

A stationary point is found by variating $||A||$ w.r.t. gauge transformations

$$\delta A_\mu = i[A_\mu,\Omega] + \partial_\mu\Omega \quad , \quad \delta||A||^2 = 2i\int tr(\partial_\mu A^\mu)\Omega d^4x = 0 \quad \forall\Omega \quad \Longleftrightarrow \quad \partial_\mu A^\mu = 0$$

Therefore in Lorentz gauge, A_μ^a contains as few pure gauge as possible, if the stationary point is a minimum. An expansion in A is thus most rapidly convergent in Lorentz gauge. In applications where A is not needed in total e.g. when only a certain momentum region is probed, different norms and different gauges may be optimal in the sense discussed above. Especially one should include derivatives of A into the norm in order to guarantee a smooth A which is important for high energies.

[1]In Euclidian space this is a positive definite norm

8.3 On the Gauge in Instanton Physics

When calculating GI quantities in the background of one instanton in a GI way the choice of gauge is only a matter of convenience. But one can see that there are large cancellations between different terms in regular gauge at large distances due to their slow decay and in singular gauge at small distances due to the topological singularity at the instanton center. For non-GI invariant quantities like the gluon or quark propagator, or when making some unsystematic approximation, the lesson is to use singular/regular gauge when dealing with low/high energies to avoid these cancellations. This is consistent with the discussion given above. Singular as well as regular gauge fulfill the Lorentz condition. $||A_{sing}||$ is finite and a minimum. A_{sing} is therefore a good choice for low energies. For high energies it is important to have a smooth A which is obviously only satisfied by the regular gauge.

To linearly superpose instantons they have to decay rapidly enough. Therefore one has to use singular gauge. This argument can in principle be circumvented by superposing two fields A_N and $A_{\overline{N}}$ the former/latter being an exact multi-instanton/anti-instanton configuration in regular gauge. Despite this, for low energies singular gauge is in any case a good choice and for high energies a one instanton approximation is already a good approximation.

8.4 The Quark Propagator in Axial Gauge *

A specific example to test the gauge dependence is the quark propagator. The contribution of one instanton of radius ρ to $M(p) = ip^2\bar{S}_I(p)$ which usually is interpreted as a constituent quark mass is shown in figure B.2 in regular, singular and axial gauge. The regular graph is larger than the singular at low momentum and the singular graph shows the slow decay (only polynomial in $1/p$) for large momenta. The analytical expressions are well known and are listed in Appendix A.2 together with expressions in axial gauge which will be derived and discussed below.

A correlator containing color-non-singlet operators can be made GI by connecting distant points with a special path-ordered exponential containing the gauge field. The exponential ensures the parallel transport of color from one point to the other. The GI quark propagator may symbolically written as

$$S_{ax}(x,y) = \langle 0|\Psi(x) P\exp\left(i\int_x^y dz\cdot A(z)\right)\bar{\Psi}(y)|0\rangle \tag{8.1}$$

P denotes path-ordering. We have already defined S_{ax} to be its color singlet part because only the singlet part is GI. S_{ax} will be called the axial propagator because in axial gauge with $n_\mu = x_\mu - y_\mu$ the exponential vanishes. In the one instanton background in zeromode approximation we get

$$S_{ax}(x,y) = \frac{\mathbb{1}_c}{N_c} tr_c\left[P\exp\left(i\int_x^y dz\cdot A(z)\right)\psi(x)\bar{\psi}(y)\right]$$

where A is now the instanton field and ψ is the zeromode in any gauge. In a coordinate system where the instanton sits at the origin and $x-y$ is in time direction[2] ($\mathbf{x}=\mathbf{y}=\mathbf{z}$)

[2] Although working in Euclidian space we will adopt the Minkowskian language $(x_0, \mathbf{x})$ = (time, space).

the path ordered exponential reduces to an ordinary exponential. Alternatively we could have tried to find a gauge transformation which transforms the regular gauge in axial gauge. In both cases we get:

$$S_{ax}(x,y) = \frac{1}{N_c} tr_c \left[\psi_{ax}(x)\bar{\psi}_{ax}(y)\right] \quad , \quad \psi_{ax}(x) = R(x)\psi_{reg}(x), \tag{8.2}$$

$$R(x) = e^{\pm i\alpha(x)\frac{\tau\mathbf{x}}{|\mathbf{x}|}} = \cos\alpha(x) \pm i\frac{\tau\cdot\mathbf{x}}{|\mathbf{x}|}\sin\alpha(x) =: \pm i\tau_\mu^\pm \cdot \tilde{x}(x) \tag{8.3}$$

$$\alpha(x) = \frac{|\mathbf{x}|}{\sqrt{\mathbf{x}^2+\rho^2}} \arctan\frac{x_0}{\sqrt{\mathbf{x}^2+\rho^2}} \tag{8.4}$$

$\alpha(x)$ may also be written in a covariant form

$$\alpha(x) = \pm\left(1 + \frac{\rho^2(x-y)^2}{x^2y^2-(xy)^2}\right)^{-1/2} \arctan\sqrt{\frac{(x^2-(xy))^2}{x^2y^2-(xy)^2+\rho^2(x-y)^2}}$$

but now $\alpha(x)$ depends also on y and the expression for the propagator no longer factorizes. The reason for this is that the axial gauge is not covariant, but the definition of the propagator is. Inserting (8.3) and (8.8) into (8.2) we get

$$S_{ax}(x,y) = \frac{1}{N_c}\left((\tilde{x}\tilde{y}) - \frac{1}{2}\tilde{x}_\mu\tilde{y}_\nu\sigma^{\mu\nu}\right)\frac{1\pm\gamma_5}{2}\varphi_{reg}(x)\varphi_{reg}(y) \tag{8.5}$$

Inserting (8.3), (8.4) and (8.8) into (8.5) the space-time averaged propagator can be expressed as an integral over elementary functions

$$\bar{S}(x-y) = \int_0^\infty dr \int_{-\infty}^\infty dt\, 4\pi r^2 \cos\left[\frac{r}{R}\left(\arctan\frac{t+|x-y|}{R} - \arctan\frac{t}{R}\right)\right]\cdot$$

$$\cdot\frac{1}{2N_c}\frac{\rho^2}{\pi^2(R^2+(t+|x-y|)^2)^{3/2}(R^2+t^2)^{3/2}} \quad , \quad R^2 = r^2+\rho^2$$

The difference between the propagator in regular and axial gauge is the insertion of the cos[...] factor. Therefore the axial propagator is everywhere smaller than the regular propagator, except at $x = y$ where they coincide because the path-ordered exponential is one. At large distances it is smaller by a factor $\pi/4$. Instead of performing the integration in coordinate space, let us go directly to the more interesting momentum space representation:

$$\bar{S}_I(p) = \frac{1}{2N_c}\varphi_{ax}^\mu(p)\varphi_{ax}^{\mu\dagger}(p) \quad , \quad \varphi_{ax}^\mu(p) = \int \tilde{x}^\mu(x)\varphi_{reg}(x)e^{ipx}dx \tag{8.6}$$

Although φ_{ax}^μ does not transform like a vector, we can choose a convenient direction of p because $\varphi\varphi^\dagger$ is a Lorentz scalar. For pure spacelike p the spatial components of φ vanish because the integrand is anti-symmetric w.r.t time reflection. Only the time component is nontrivial

$$\varphi_{ax}(p) := \varphi_{ax}^0(p) = \int d^3r \int dt\, \cos\left[\frac{r}{R}\arctan\frac{t}{R}\right]\frac{\rho}{\pi(R^2+t^2)^{3/2}}e^{i\mathbf{p}\cdot\mathbf{r}}$$

With the following hints

$$
\begin{aligned}
\cos(\gamma \arctan x) &= \mathrm{Re}\left(\frac{1+ix}{1-ix}\right)^{\gamma/2} \\
\int_{-\infty}^{\infty}(R-it)^{-\alpha}(R+it)^{-\beta}dt &= 2\pi(2R)^{1-\alpha-\beta}\frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)\Gamma(\beta)} \\
\Gamma(\frac{3}{2}-x)\Gamma(\frac{3}{2}+x) &= \frac{(1/4-x^2)\pi}{\cos\pi x} \\
\int d^3r\, e^{i\mathbf{p}\cdot\mathbf{r}} f(r) &= \frac{2\pi}{p}\int_0^\infty f(r)\sin(pr)r\,dr
\end{aligned}
\tag{8.7}
$$

the reader should be able to perform the t and the angular integration $d\Omega_r$,

$$
\varphi_{ax}(p) = \frac{8}{p\rho}\int_0^\infty \cos\left(\frac{\pi r}{2R}\right)\sin(pr)r\,dr \tag{8.8}
$$

I was not able to perform this last integral analytically, but for small momenta it is easy to see that $\varphi^0_{ax}(p)$ behaves like $\pi^2\rho/p$. For large p it decays like $\sim e^{-p\rho}$ with a non-polynomial coefficient because of an essential singularity at $r = \pm i\rho$. Comparing φ_{reg}, φ_{sing} and φ_{ax} plotted in figure B.2 we see that the axial φ lies somewhat in between the regular and the singular. So one may conclude that axial gauge is a good compromise for all momenta.

The calculation of the GI propagator seems to make the discussion of its gauge dependent partners obsolete. I will now argue that this is not the case. The reason is that there are a huge number of GI definitions of a quark propagator and (8.1) is only one possible choice. One obvious generalization is to choose a more complicated path from x to y than a straight line. The next thing one could do is not to restrict oneself to a specific path, but to take into account all paths one is interested in and average the results with arbitrary weights. Another possibility is to let the path depend on the gauge field itself, as long as this choice is made in a GI way. Finally one can combine both generalizations. I am sure that it is possible to produce any result for the propagator with a suitable generalized definition. The advantage of the standard axial propagator is, that the definition is simple and that the non local operator has a physical interpretation. It creates a quark-antiquark pair connected by a thin gluon flux tube. This might be a good choice for a non-local meson creation operator. But it is also plausible that one of the generalizations given above is even better. The only thing I want to point out is, that the GI definition for the propagator given above is nothing more than to work in axial gauge. One still has to choose the right gauge using more sophisticated arguments.

8.5 Effective Quark Mass *

The only reason for performing the $N_c \to \infty$ limit is to make the N_c dependence of the resulting formulas simple. The accuracy has been checked to be within the standard 10% for $N_c = 3$ usually achieved by $1/N_c$ expansion. Here the accuracy can simply be understood. The actual expansion parameter is not $1/N_c = 1/3$ itself but $1/b \approx 1/11$. The following asymptotic formulas will be used

$$N_c!^{1/N_c} \doteq N_c/e \quad , \quad b \doteq \frac{11}{3}N_c \quad , \quad C_{N_c}^{1/b} \doteq 2.22b^{-6/11}$$
$$\rho^5 D(\rho) \sim (2.22(S_0/b)^{6/11}\rho\Lambda)^b \quad , \quad S_0/b = \frac{24\pi^2}{11}(g_0^2 N_c)^{-1} \tag{8.9}$$

Every equality in the large N_c limit will be marked with a dot. Notice that in this limit instantons of size $\rho < \frac{1}{2.22}(S_0/b)^{-6/11}\Lambda^{-1}$ are completely suppressed. Above this threshold the instanton density gets infinite. S_0/b is independent of N_c because the coupling is $g_0 \sim 1/\sqrt{N_c}$. Λ is the QCD scale.

In the presence of one light quark flavor the instanton density $D(\rho)$ has to be multiplied with the functional determinant of the Dirac operator

$$Det(i\not{D} + im) \approx 1.34m\rho$$

which is proportional to m because of a zeromode of $\not{D}$. The quark propagator in the background of one instanton is dominated by this zeromode

$$S_I(p,q) = \frac{\psi_I(p)\psi_I^\dagger(q)}{im}$$

Averaging this expression over all collective coordinates γ_I one gets

$$M(p) := ip^2\bar{S}_I(p) = \frac{1.34}{2N_c}\int_0^\infty d\rho\, p^2\rho D(\rho)\varphi^2(p)$$

Summing the contribution to the propagator of $0, 1, 2, 3, \ldots$ instantons, which is the analog of a selfenergy resummation in perturbation theory,

$$S(p) = \frac{1}{\not{p}} + \frac{1}{\not{p}}\frac{M(p)}{i}\frac{1}{\not{p}} + \frac{1}{\not{p}}\frac{M(p)}{i}\frac{1}{\not{p}}\frac{M(p)}{i}\frac{1}{\not{p}} + \ldots = \frac{1}{\not{p} + iM(p)}$$

justifies to call $M(p)$ a dynamical quark mass. Expressions of φ in various gauges are given in appendix A.2. The graphs of $\varphi(p)$ in singular and regular gauge cross over at

$$\varphi_{sing}(p) = \varphi_{reg}(p) \quad \Longleftrightarrow \quad 2p\rho \approx 2.5$$

Therefore one should use regular gauge for large ρ and singular gauge for small ρ. This choice of gauge also makes the integral convergent for large ρ. At this stage we have no infrared problem. Using regular gauge in the whole integration interval we get

$$M_{reg}(p) = Bp(\frac{\Lambda}{2p})^b \quad , \quad B = 1.34 \cdot 16\pi^2(C_{N_c}b^{2N_c})(S_0/b)^{6b/11}I_b/N_c$$

$$I_b = \int_0^\infty dz\, z^{b-2}e^{-z} = (b-2)! \quad , \quad z = 2p\rho$$

The integral is sharply dominated by

$$z \doteq b \pm b^{1/2} >> 2.5$$

therefore the result is independent of the choice of gauge for $z < 2.5$ justifying our use of regular gauge over the whole integration interval. Axial gauge would lead to nearly the same result as can be seen from figure B.2. Using singular gauge for large ρ would produce a divergent integral dominated by arbitrary large instantons inconsistent with the choice of gauge discussed above. The infrared "problem" shows up in the rapid raise of $M(p)$ for low p, which effectively suppresses the propagation of quarks with low virtuality p^2. Consider some process involving quarks at distances $x = 1/p_c$. The effective quarkmass $M(1/x)$ is dominated by instantons of much larger size

$$\rho = \rho_c(1 \pm b^{-1/2}) >> x \quad , \quad \rho_c = \frac{b}{2p_c}.$$

In other words, given an instanton of radius ρ influences the physics at a much smaller scale $x = \frac{2}{b}\rho << \rho$. Therefore the interior of the instanton is probed and one should avoid the singularity at its center by using regular gauge.

8.6 The Quark Condensate

Let us now calculate a real physical gauge invariant observable, the quark condensate

$$\langle\bar{\psi}\psi\rangle := \lim_{x\to 0} \mathrm{tr}_{CD}(S(x) - S_0(x)) = -4iN_c \int \frac{M(p)}{p^2 + M^2(p)} \frac{d^4p}{(2\pi)^4}$$

Inserting M(p) and performing the angular integration we get

$$|\langle\bar{\psi}\psi\rangle| = \frac{N_c}{16\pi^2} B^{3/b} J_b \Lambda^3$$

$$J_b = \int_0^\infty \frac{z^{b+2}}{1+z^{2b}} dz = \frac{\pi}{2b\sin(\frac{b+3}{2b}\pi)} \doteq \frac{\pi}{2b} \quad , \quad p = B^{1/b}\frac{\Lambda}{2}z$$

The integral is finite and sharply dominated by $z \doteq 1 \pm b^{-1}$. Without resummation of the selfenergies the integral J_b and thus condensate would have turned out to be infinite. The condensate is dominated by quark wavefunctions with momenta

$$p = p_c(1 \pm b^{-1}) \quad , \quad p_c = \beta b \frac{\Lambda}{2} \quad , \quad \beta := \frac{1}{b} B^{1/b} \doteq \frac{2.22}{e} (S_0/b)^{6/11}$$

and depends on Λ and g_0

$$|\langle\bar{\psi}\psi\rangle|^{1/3} = 0.139\beta b\Lambda \quad .$$

Expressing p_c and ρ_c in terms of $|\langle\bar{\psi}\psi\rangle|$ by eliminating β we get our main result

$$\begin{aligned} p_c &= 3.59|\langle\bar{\psi}\psi\rangle|^{1/3} \\ \rho_c &= \frac{1.96}{N_c}|\langle\bar{\psi}\psi\rangle|^{1/3} \\ 2.22N_c\Lambda &= (g_0^2 N_c)^{6/11}|\langle\bar{\psi}\psi\rangle|^{1/3} \end{aligned} \tag{8.10}$$

A weak point is the experimental extraction of g_0. It should be extracted from a reliable treelevel process at low energies presumably of the order of ρ_c. In QCD improved Bag-Models the main nonperturbative effect is modeled by the bag and the hyperfinesplitting is caused by a one gluon exchange. g_0 extracted from $\Delta - N$ splitting is [10]

$$g_0^{bag} \approx 2.6$$

Let me also give a theoretical guess of g_0. The change to a two loop expression for the instanton density $S_{0/1} \rightsquigarrow S_{1/2}$ can be effectively performed by only replacing S_0 in the following way

$$(S_0/b)^{6/11} \rightsquigarrow (\ln \frac{1}{\rho\Lambda})^\alpha \quad , \quad \alpha = \frac{15}{121}$$

Because α is very small $(\ln \frac{1}{\rho\Lambda})^\alpha$ is approximately one in a large range of values for $\rho\Lambda$. and for

$$g_0^{guess} = 2.7\sqrt{3/N_c}$$

the two 2 loop density coincides with the one loop density. Because we do not believe that the 2 loop density is an improvement, one should not take g_0^{guess} too seriously. At least it is not in contradiction with g_0^{bag}.

The condensate is well known to be $|\langle\bar{\psi}\psi\rangle|^{1/3} = 240\text{MeV}$. Setting $N_c = 3$ and taking $g_0 = 2.6$ for granted we get

$$\begin{aligned} p_c \pm \Delta p &= (860 \pm 80)\text{MeV} \\ \rho_c \pm \Delta\rho &= (160 \pm 50)^{-1}\text{MeV} \\ \Lambda_{PV} &\approx 190\text{MeV} \end{aligned} \tag{8.11}$$

The most interesting thing is, that the condensate is sharply dominated by quark field wave functions of rather large momentum p_c. On the other hand the dominating instantons have a very large radius ρ_c, 4 times larger than usually assumed in instanton liquid models. Nevertheless the predicted value of Λ_{PV}, which of course must be assigned a large error because of the rough estimate of g_0, is in agreement with experiment.

8.7 Summary

Whenever one is calculating gauge dependent objects or when making gauge breaking approximations, one is confronted with the problem of choosing a "good" gauge. Specializing the general discussion of Section 8.2 to the case of instantons, we came to the conclusion that the regular gauge is appropriate for small distances and the singular gauge for processes involving large distances. The GI propagator was defined, calculated and compared to the propagator in singular and regular gauge (figure B.2). The conclusion was, that the GI propagator is not a-priori a good choice, but lies somewhat in between regular and singular gauge.

Using an appropriate gauge along the lines discussed in Section 8.2 we were able to derive a finite quark condensate without taking an infrared cut-off for the instanton radius nor relying on some instanton model. The linear relation between $|\langle\bar{\psi}\psi\rangle|^{1/3}$ and the QCD scale Λ is in agreement with experiment. The condensate if formed by quark fields of high momenta $p_c = 860$MeV mainly lying within the sharp region $\Delta p = 80$MeV. The dominating instantons are very large ($\rho_c = 160$MeV).

Chapter 9

Conclusions

9.1 Summary

Some known and various new results have been obtained in this work. In most cases, the computation was based on the Instanton Liquid Model.

The following new insights were gained.

- For the quark propagator dynamical quark loops can be absorbed in a renormalized instanton density, which can be identified with the gluon condensate (Chapter 3).
- Concise Bethe-Salpeter equations have been obtained and were solved for the quark 4-point functions. In the flavor singlet channel, a chain of quark loops contribute, which is responsible for the missing $U(1)$ Goldstone bosons (Chapter 4).
- From the meson correlators the masses and couplings of the σ, ρ, ω, a_1 and f_1 mesons have been obtained by a spectral fit (Table 5.3, Figures B.3-B.8).
- The η' mass has been predicted successfully (Equation 6.16).
- The axial proton form factors of the singlet current $j_{\mu 5}$, the gluon current K_μ, and the anomaly a have been computed and their gauge (in)dependence has been discussed (Table 6.4). It has been shown that $A(0) = -1$ is independent of the number of flavors.
- For small momentum a ghost and a gauge independent gluon mass have been calculated (Equations 7.32 and 7.33).
- General rules regarding the choice of a gauge, especially when to chose the singular and when the regular gauge have been established (Sections 8.1, 8.2 and 8.3).
- A gauge invariant quark propagator in the 1-instanton background has been derived (Equations 8.6 and 8.8).
- The quark condensate has been identified with the QCD scale Λ, where neither an infrared cutoff, nor a specific instanton model was necessary (Equation 8.10).

9.2 Outlook

The Instanton Liquid Model has proven to be successful in various sectors of QCD as the results of this and other works show. Although this model cannot strictly be derived from first principles, its success proves that there is some truth to it after all. The limits of this model also show up clearly: Confinement cannot be explained and the axial singlet channel still causes problems, although instantons explicitly break the $U(1)_A$ symmetry.

Future work should take into account non-zeromodes, which become important in the case of strange quarks and probably essential in respecting the axial Ward identities. Baryon correlators to determine Baryon masses could be calculated. Both extensions have already been realized numerically [28].

Although a direct calculation of the axial singlet correlators was not successful, refined arguments allowed determining the η' mass. A similar consideration may also solve the proton spin problem.

Gauge invariant glueball corrections should be calculated.

The most urgent theoretical problem is still a correctness proof of the Instanton Liquid Model. This would clarify the range of validity and increase the general acceptance of this model.

QCD is, indeed, a tough theory. Experimental physicists do a good job in measuring the hadron parameters, whereby the precision of these experiments increases steadily. This progress should challenge the theorist to keep up with them.

9.3 Acknowledgements

First and foremost I would like to thank Prof. H. Fritzsch for his willingness to take on a novice in his department. He opened the door to the fascination of particle physic and enabled contact with important particle physicists, especially Prof. Karliner in Israel, Prof. S. Veneziano, S. Forte and many others at CERN. I owe Prof. Dyakonov special thanks for many precious discussions and his prompt willingness to answer all questions via email. I wish to thank my good friend Michael Birkel for bringing fresh wind to the Institute and Dr. A. Blumhofer for many long, stimulating evenings spent at the University, during which the borderline between physics and philosophy was often crossed. Thanks also to all colleagues of the department, for occasional discussions. Thank you to my father, and M. Matuschek, and D. Holtmannspötter for proof-reading my thesis. I am also very grateful to the German "Forschungs-Gemeinschaft" for financial support.

Appendix A

A.1 Notations

Nearly all occurring factors 2π can be absorbed in the following definitions:

$$\bar{\delta}^d(\cdots) := (2\pi)^d \delta(\cdots) \quad , \quad \int \bar{d}^d p := \int \frac{d^d p}{(2\pi)^d} \quad ,$$

At many places vectors and operators in Dirac notation in $I\!R^4$ are used:

$$\langle x|p\rangle = e^{ipx} \quad , \quad \langle x|y\rangle = \delta^4(x-y) \quad , \quad \langle p|q\rangle = \bar{\delta}^4(p-q)$$

$$\langle x|\psi\rangle = \psi(x) \quad , \quad \langle p|\psi\rangle = \hat{\psi}(p) \quad , \quad \langle p|S|q\rangle = S(p,q)$$

$$\int \bar{d}^4 p \, |p\rangle\langle p| = 1\!\!1 \quad , \quad \int d^4 x \, |x\rangle\langle x| = 1\!\!1 \quad ,$$

$$[\hat{p}, X] = i(\partial_\mu X)$$

This notation should not be confused with the vacuum state $|0\rangle$ and the proton state $|ps\rangle$ of the QCD Hilbert space and the averaging $\langle\cdots\rangle_I$ over instantons.

$$\begin{aligned}
N_c &= \text{Number of colors} \\
N_f &= \text{Number of quark flavors} \\
m &= \text{Current masses} \\
M &= \text{Dynamic masses} \\
\mathrm{tr}_D &= \text{Trace in Dirac space} \\
\mathrm{tr}_C &= \text{Trace in color space} \\
\mathrm{Tr} &= \text{Functional trace} \\
\mathrm{Det} &= \text{Functional determinant} \\
\lambda^a/2 &= \text{Generators of } SU(N_c), \quad a = 1\ldots N_c^2-1 \\
\tau^a/2 &= \lambda^a/2 = \text{Generators of } SU(2)_c, \quad a = 1\ldots 3
\end{aligned}$$

A.2 Instantons in Singular, Regular and Axial Gauge

Instantons are solutions of the classical Yang-Mills equations of motion. The instanton at the origin in standard orientation in singular, regular and axial gauge is given by:

$$\begin{aligned} A_\mu^{sing}(x) &= \eta^{\pm}_{\mu\nu}\frac{x_\nu}{x^2}\frac{\rho^2}{x^2+\rho^2} \quad , \quad \tau^\pm_\mu\tau^\mp_\nu = \delta_{\mu\nu} + i\eta^\pm_{\mu\nu} \\ A_\mu^{reg}(x) &= \eta^{\mp}_{\mu\nu}\frac{x_\nu}{x^2+\rho^2} \quad , \quad \tau^\pm_\mu = (\pm i, \tau) \\ A_\mu^{ax}(x) &= R(x)A_\mu^{reg}(x)R^\dagger(x) + iR(x)\partial_\mu R^\dagger(x) \end{aligned} \tag{A.1}$$

The upper/lower sign denotes an instanton/antiinstanton ($Q = \pm 1$).

$$R(x) = \pm i\tau^\pm_\mu \tilde{x}^\mu(x) \quad , \quad \tilde{x}^\mu(x) = \begin{pmatrix} \cos\alpha(x) \\ \frac{\mathbf{x}}{|\mathbf{x}|}\sin\alpha(x) \end{pmatrix}$$

$$\alpha(x) = \frac{|\mathbf{x}|}{\sqrt{\mathbf{x}^2+\rho^2}}\arctan\frac{x_0}{\sqrt{\mathbf{x}^2+\rho^2}}$$

The covariant derivative $\not{D}$ has a zeromode

$$i\not{D}\psi = (i\not{\partial} - \not{A})\psi = 0$$

where the zeromode has the following form:

$$\begin{aligned} \psi_{sing}(x) &= \sqrt{2}\varphi(x)\frac{\not{x}}{|x|}\chi \\ \psi_{reg}(x) &= \sqrt{2}\varphi(x)\chi \quad , \quad \varphi(x) = \frac{\rho}{\pi(x^2+\rho^2)^{3/2}} \\ \psi_{ax}(x) &= \sqrt{2}\varphi(x)R(x)\chi \end{aligned} \tag{A.2}$$

χ is a color & Dirac spinor, given by

$$\chi^\pm\bar{\chi}^\pm = \frac{1}{16}\gamma_\mu\gamma_\nu\frac{1\pm\gamma_5}{2}\tau^\mp_\mu\tau^\pm_\nu \quad .$$

For light quarks the propagator is dominated by the zeromode. When averaged over the instanton orientation, position and charge the propagator is diagonal in momentum space and given by

$$\langle\psi(p)\psi^\dagger(p)\rangle = \frac{1}{2N_c}\varphi^2(p),$$

where

$$\begin{aligned} \psi_{sing}(p) &= \sqrt{2}\varphi_{sing}(p)\frac{\not{p}}{|p|}\chi \quad , \quad \varphi_{sing}(p) = \pi\rho^2\frac{d}{dz}[I_1(z)K_1(z) - I_0(z)K_0(z)]_{z=p\rho/2} \\ \psi_{reg}(p) &= \sqrt{2}\varphi(p)\chi \quad , \quad \varphi_{reg}(p) = \frac{4\pi\rho}{p}e^{-p\rho} \\ \psi_{ax}(p) &= FT\{\psi_{ax}\}(p) \quad , \quad \varphi_{ax}(p) = \frac{8}{p\rho}\int_0^\infty \cos\left(\frac{\pi r}{2\sqrt{r^2+\rho^2}}\right)\sin(pr)r\,dr \end{aligned} \tag{A.3}$$

$\frac{p}{\rho}\varphi(p)$	singular	regular	axial
$p\rho \ll 1$	2π	4π	π^2
$p\rho \gg 1$	$\frac{12\pi}{(p\rho)^3}$	$4\pi e^{-p\rho}$	$\sim e^{-p\rho}$

Table A.1: Asymptotic behaviour of $\frac{p}{\rho}\varphi(p)$

$I_\nu(z)$ and $K_\nu(z)$ are modified Bessel functions. The asymptotics are given in the following table

The constituent mass of a quark in the Instanton Liquid Model is

$$M_\rho(p) \sim p^2\varphi^2(p) \quad .$$

$M_\rho(p)$ is plotted in figure B.2 in all three gauges.

A.3 Averaging over the Instanton Parameter γ_I

The instanton in general orientation and location and the corresponding zeromode have the following form

$$\begin{aligned} A_{I\mu}(x) &= O_I^{ab}A_\mu^b(x-z_I) \\ \psi_I(x) &= U_I\psi(x-z_I) \quad , \quad \chi_I := U_I\chi \quad , \quad \frac{1}{2}\mathrm{tr}(U\lambda^a U^\dagger\lambda^b) = O^{ab} \\ \gamma_I = (z_I, O_I, \rho_I, Q_I) &= \text{(location, orientation, radius, topological charge)} \end{aligned}$$

A_μ^b is and instanton in standard orientation with radius $\rho = \rho_I$ and charge $Q_I = \pm 1$ and ψ is the corresponding zeromode, given in Appendix A.2. $U_I \in SU(N_c)$ and $O_I \in Ad[SU(N_c)]$ are orientation matrices in fundamental and adjunct representation, respectively.

t various points I have to average over the collective coordinates γ_I:

$$\langle \ldots \rangle_I = \int d\gamma_I\, D(\rho_I)\ldots = \frac{1}{2}\sum_{Q_I=\pm 1}\int d^4z_I dO_I d\rho_I\, D(\rho_I)\ldots \tag{A.4}$$

The following assumption on D defines the Instanton Liquid Model:

$$D(\rho_I) = n\delta(\rho_I - \rho) \quad , \quad n = (200\ \mathrm{MeV})^4 \quad , \quad \rho = 600\ \mathrm{MeV}^{-1} \quad . \tag{A.5}$$

The most important formulas for the Haar measure $\int dO_I$ and $\int dU_I$ are:

$$\int dO\ 1 = 1 \quad , \quad \int dO\ O^{ab}O^{cd} = \frac{1}{N_c^2-1}\delta^{ac}\delta^{bd} \tag{A.6}$$

$$\int dU\ 1 = 1 \quad , \quad \int dU\ U_k^i U_l^{\dagger j} = \frac{1}{N_c}\delta_l^i\delta_k^j$$

Integrals over an odd number of matrices are zero. The following formulas are also useful

$$N_C \langle \chi_I^{\pm} \bar{\chi}_I^{\pm} \rangle_I = \mathrm{tr}_C \chi_I^{\pm} \bar{\chi}_I^{\pm} = \frac{1}{2}(\frac{1 \pm \gamma_5}{2}) \tag{A.7}$$

where the averaging over Q_I has not been performed yet.

$$\bar{\chi}\chi = \mathrm{tr}_{CD} \chi\bar{\chi} = 1 \tag{A.8}$$

Collecting space, color and Dirac terms yields

$$\psi_I^\dagger(p)\psi_I(p) = 2\varphi^2(p) \quad , \quad \int đ^4p\, \psi_I^\dagger(p)\psi_I(p) = 1 \tag{A.9}$$

A.4 Numerical Evaluation of Integrals

The integral expressions for the meson correlators have been evaluated numerically. Two types of operations have to be performed:

1. Convolution of Lorentz covariant functions ($F_{0/5}, \Gamma_\Gamma$)
2. Fourier transformation (FT) of the correlators to coordinate space

Let us first consider the FT generalized to d dimensions:

$$\hat{f}_{\mu_1 \ldots \mu_n}(x) = F_d\{f_{\mu_1 \ldots \mu_n}\}(x) = \int đ^dp\, e^{-ipx} f_{\mu_1 \ldots \mu_n}(p) \tag{A.10}$$

For a scalar spherically symmetric function $f = f(|p|)$ the FT reduces to a one dimensional integral

$$F_d\{f(|p|)\}(x) = \int_0^\infty \left(\frac{m}{2\pi|x|}\right)^{d/2} f(m) J_{d/2-1}(m|x|)|x|\, dm \tag{A.11}$$

where J_ν are Bessel functions.

If x is not too large and f decays rapidly, the integration can be performed with Gaussian (or other) integration methods. If the decay is too slow one has to subtract the asymptotic part from f thus improving convergence. The FT of the asymptotic part can be performed analytically and has to be added to the numerical FT of the reduced function.

The FT of a general Lorentz covariant function can also be reduced to (A.11) with (formally) an increased dimension d:

$$\begin{aligned} F_d\{p_\mu f(|p|)\}(x) &= 2\pi i x_\mu F_{d+2}\{f\}(x) \qquad \text{(A.12)} \\ F_d\{p_\mu p_\nu f(|p|)\}(x) &= \frac{1}{d-1}\Big[(\delta_{\mu\nu} - \frac{x_\mu x_\nu}{x^2}) F_d\{p^2 f(|p|)\}(x) - \\ & (\delta_{\mu\nu} - d\frac{x_\mu x_\nu}{x^2})(4\pi F_{d+2}\{f\}(x) - 4\pi^2 x^2 F_{d+4}\{f\}(x))\Big] \\ & \ldots \end{aligned}$$

A.5 Numerical Evaluation of the Convolution

The convolution integral is defined as

$$f * g(p) = \int đ^d q\, f(q) \cdot g(p-q) \tag{A.13}$$

This type of integral can be reduced to the FT discussed above:

$$f * g(p) = F_d^{-1}\{F_d\{f\} \cdot F_d\{g\}\} \tag{A.14}$$

This is a quick and easy method for evaluating convolution integrals. For the disconnected part of the correlators it has the advantage that in coordinate representation F_d^{-1} can be dropped. The disadvantage of this formula is that the FTs involve oscillating integrals which are numerically problematic. If the back-transformation F_d^{-1} is needed as in the case of $F_{0/5}$ and Γ_Γ it is better to perform the convolution directly. Similar to the FT the convolution can be reduced to the scalar case. The convolution of two scalar functions can further be reduced to a two dimensional integral:

$$f * g(p) = \frac{(d/2-1)!}{2\pi^{d/2+1}(d-2)!} \int_0^\infty dr \int_0^\pi d\theta\, f(r) g(\sqrt{p^2 - 2|p| r \cos\theta + r^2})(r \sin\theta)^{d-2} r \tag{A.15}$$

It is again evaluated with Gaussian integration methods. The second advantage is that there are no problems with slowly decaying functions. Sometimes there are large cancellations between different terms. In this case it is essential to use nonadaptive integration methods because they will not result in a loss of accuracy.

The explicit reduction of the various correlators to the basic forms (A.11) and (A.15) is more or less trivial. The selfconsistency equation has been solved by iteration. The results are plotted in figure B.3 - B.8.

Appendix B

Figures

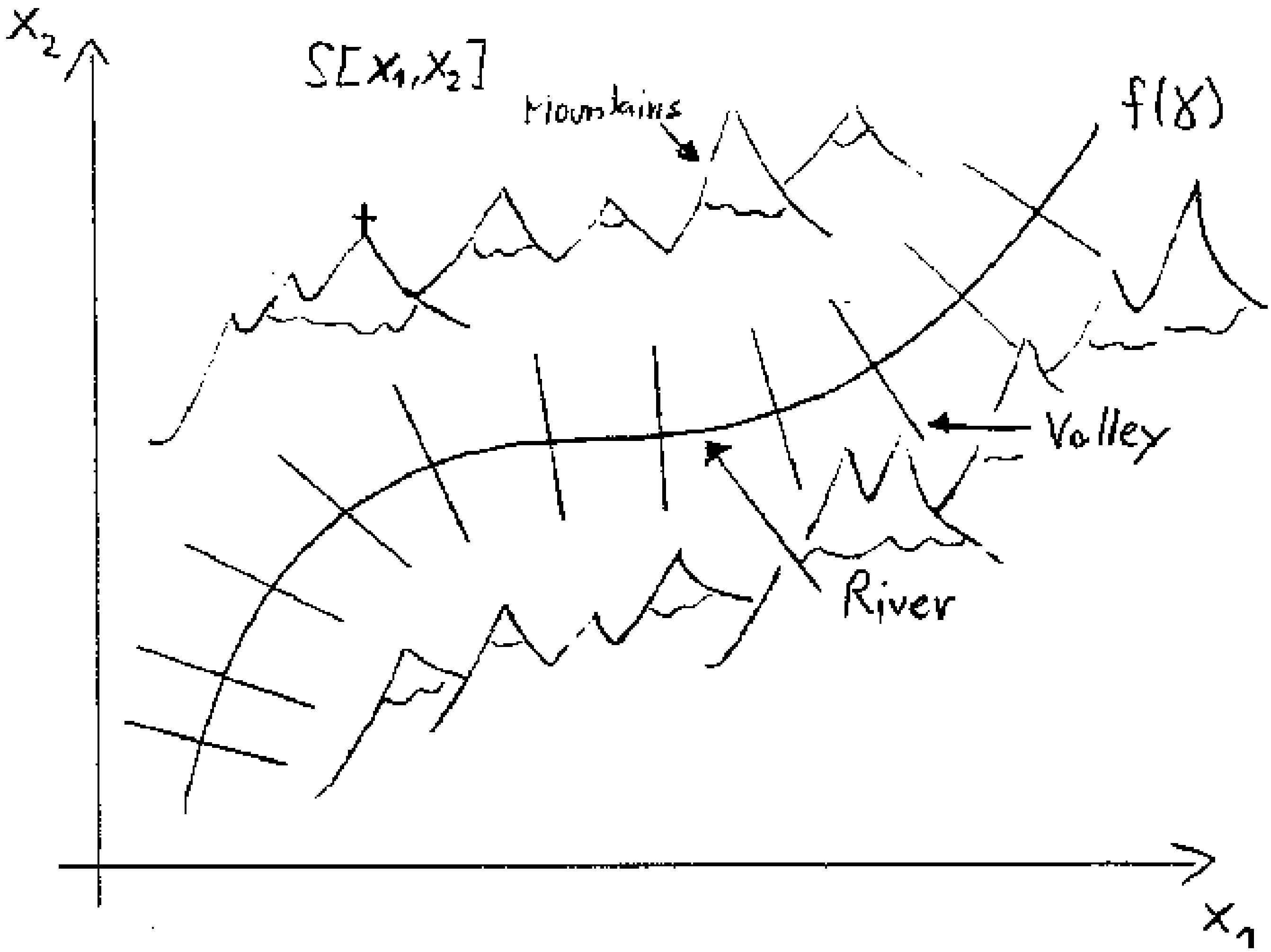


Figure B.1: *S is minimal at the river, small in the valley and large in the mountains, Therefore* $Z = \int dx_1 dx_2 \, e^{-S[x_1,x_2]}$ *is dominated by the valley.*

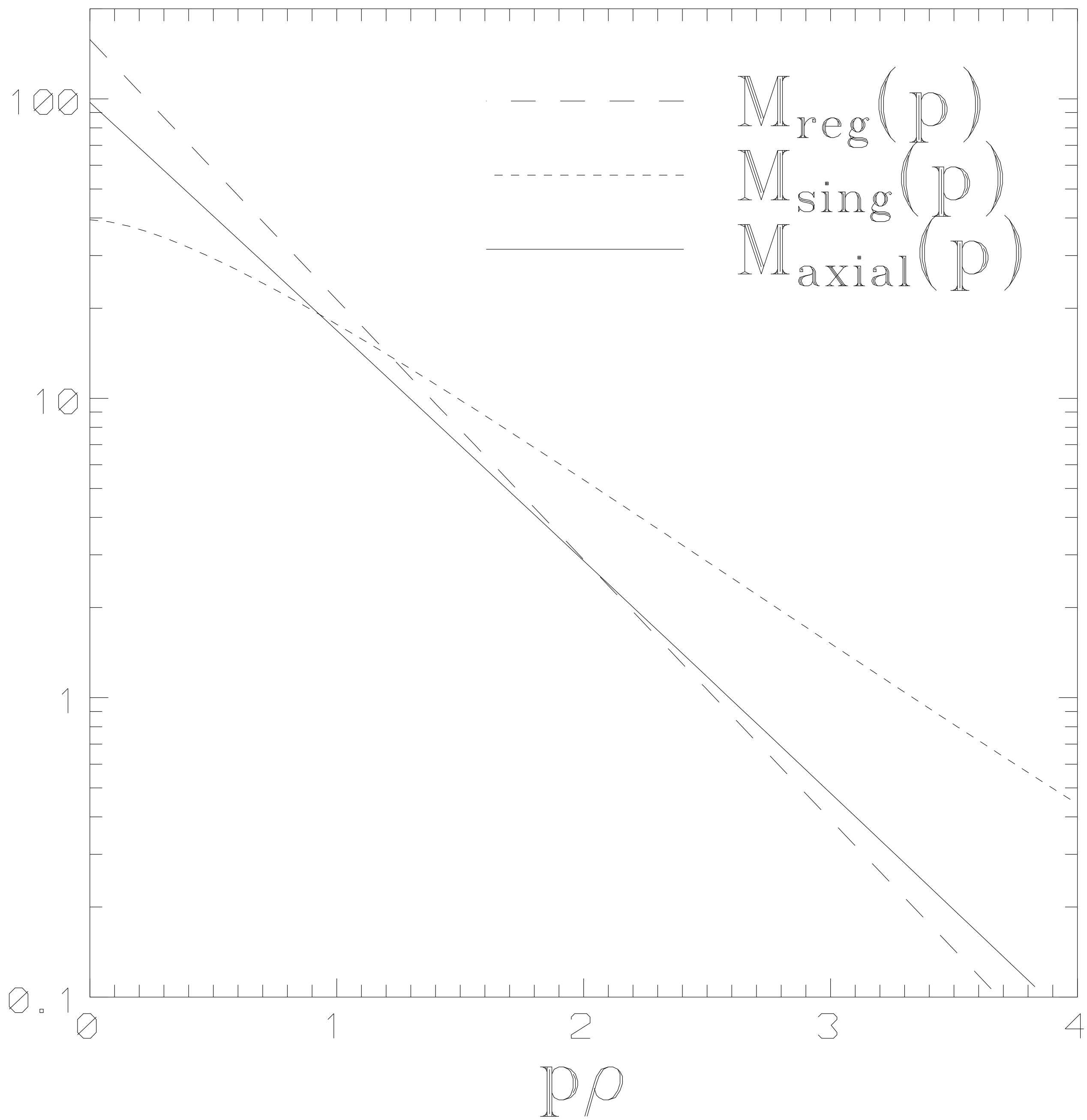


Figure B.2: *Constituent quark mass $M(p) \sim p^2\varphi^2(p)$ in singular, regular and axial gauge for fixed instanton radius ρ in arbitrary normalization. For a given momentum the corresponding lowest curve may be interpreted as the "most physical" one.*

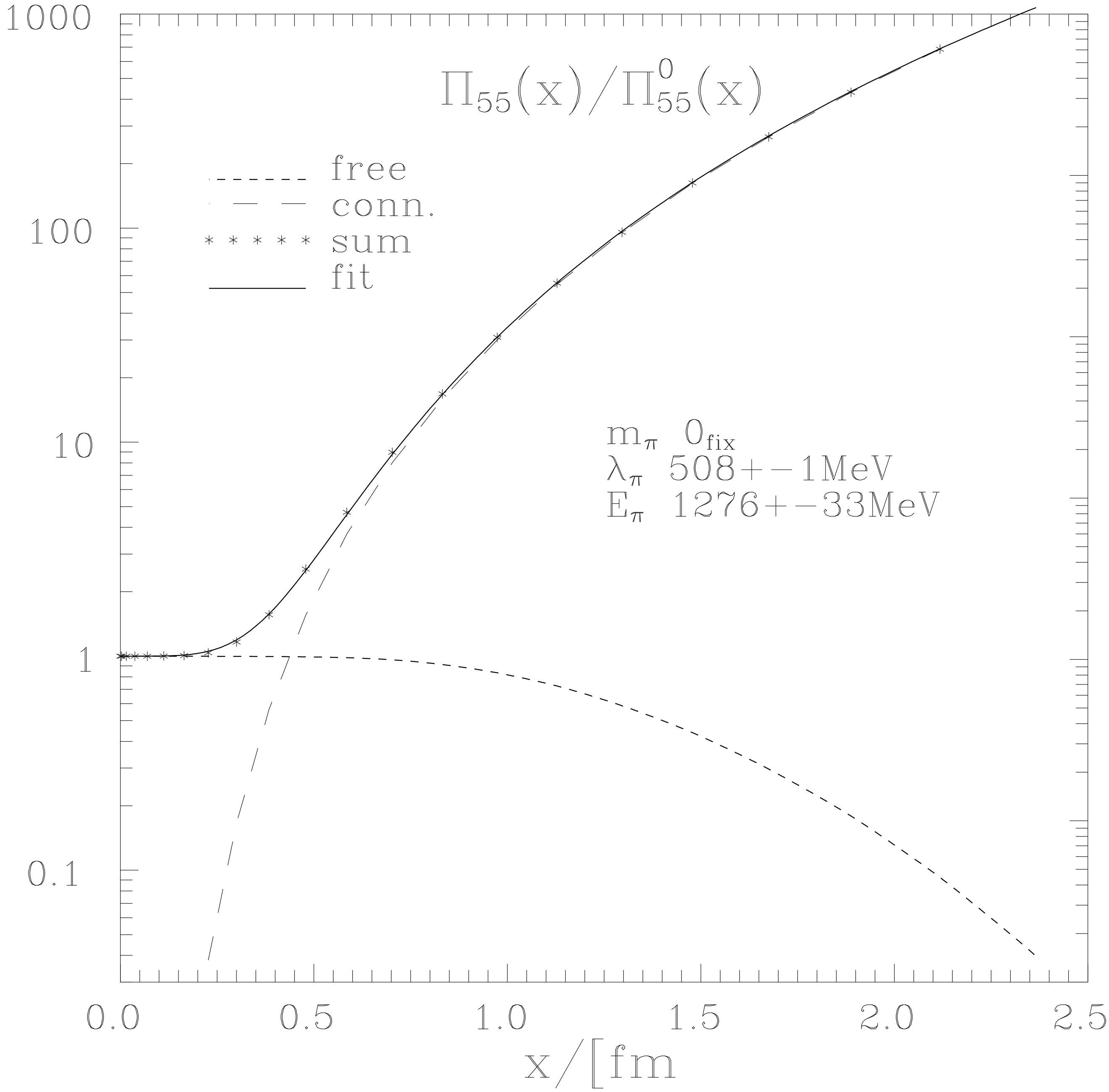


Figure B.3: *Pseudoscalar triplet correlator normalized to the free massless quark correlator. The pion coupling constant* λ_π *and the continuum threshold* E_π *are fitted in order to match the spectral ansatz with the theoretical sum of the free and the connected part.*

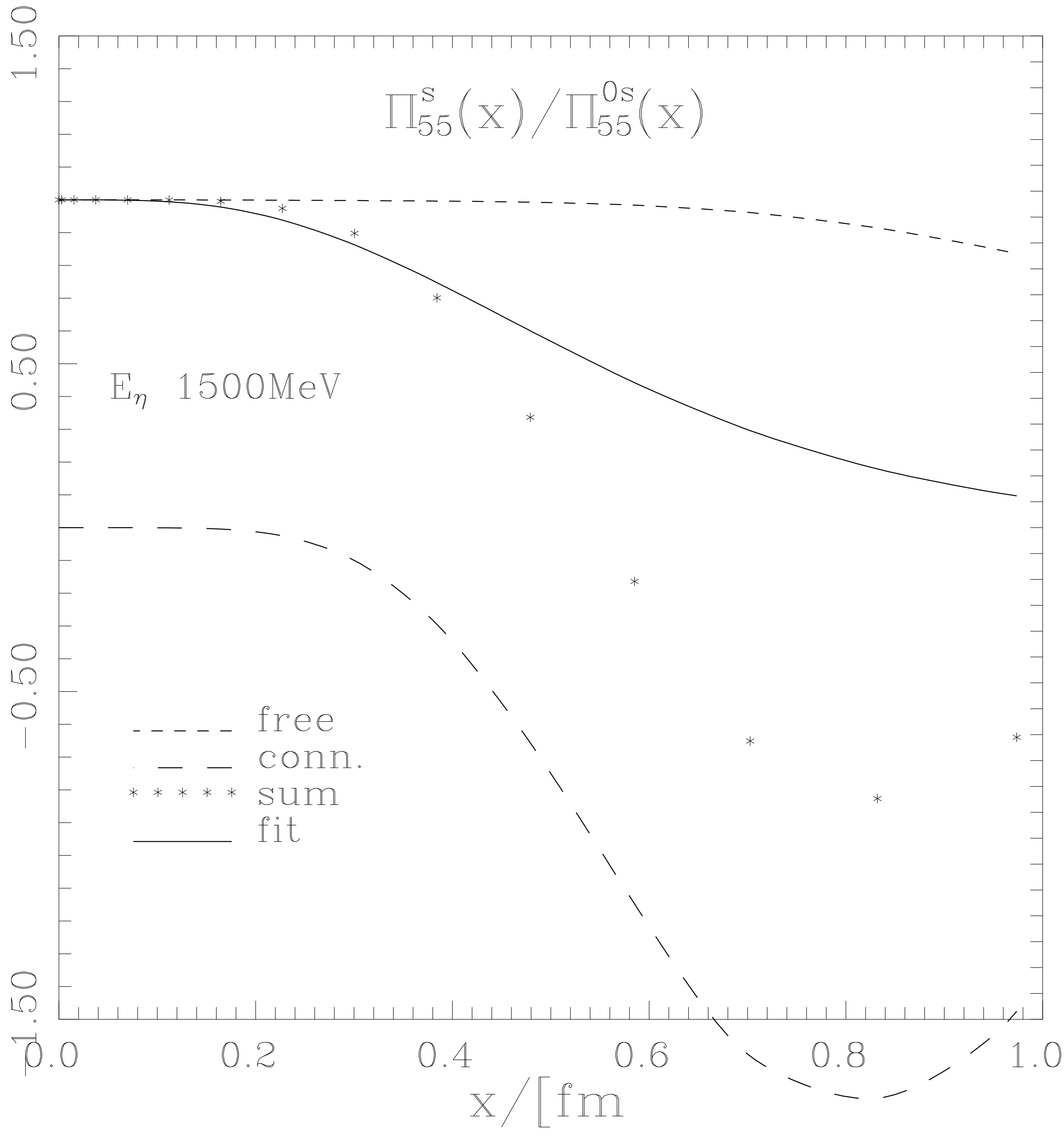


Figure B.4: *Pseudoscalar singlet correlator normalized to the free massless quark correlator. There is a strong repulsion in this channel and no boundstate is formed. The theoretical curve is compared to a curve obtained from a pure continuum spectrum above* $E_{\eta'}$.

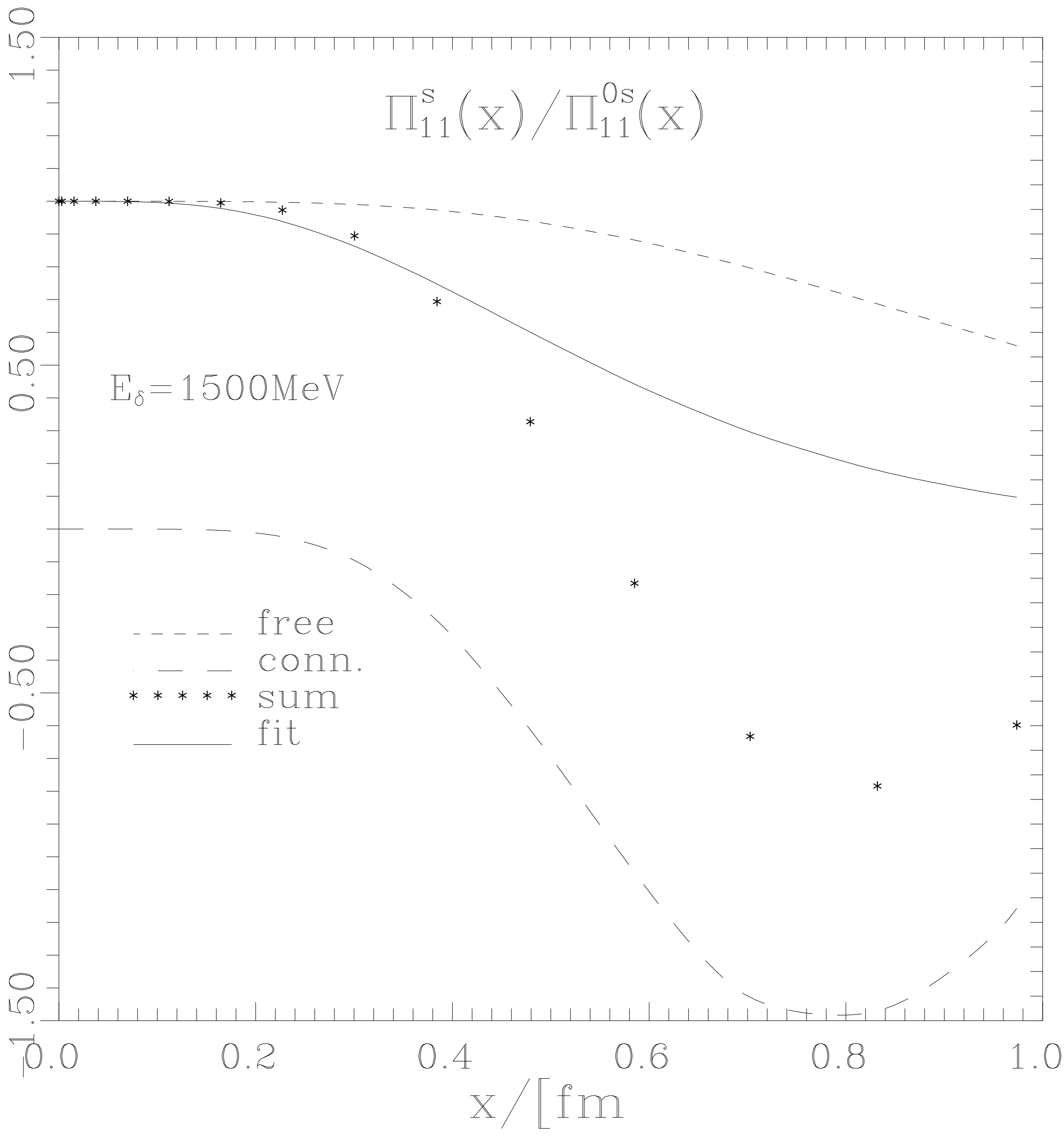


Figure B.5: *Scalar triplet correlator normalized to the free massless quark correlator. There is a strong repulsion in this channel and no boundstate is formed. The theoretical curve is compared to a curve obtained from a pure continuum spectrum above* E_δ*.*

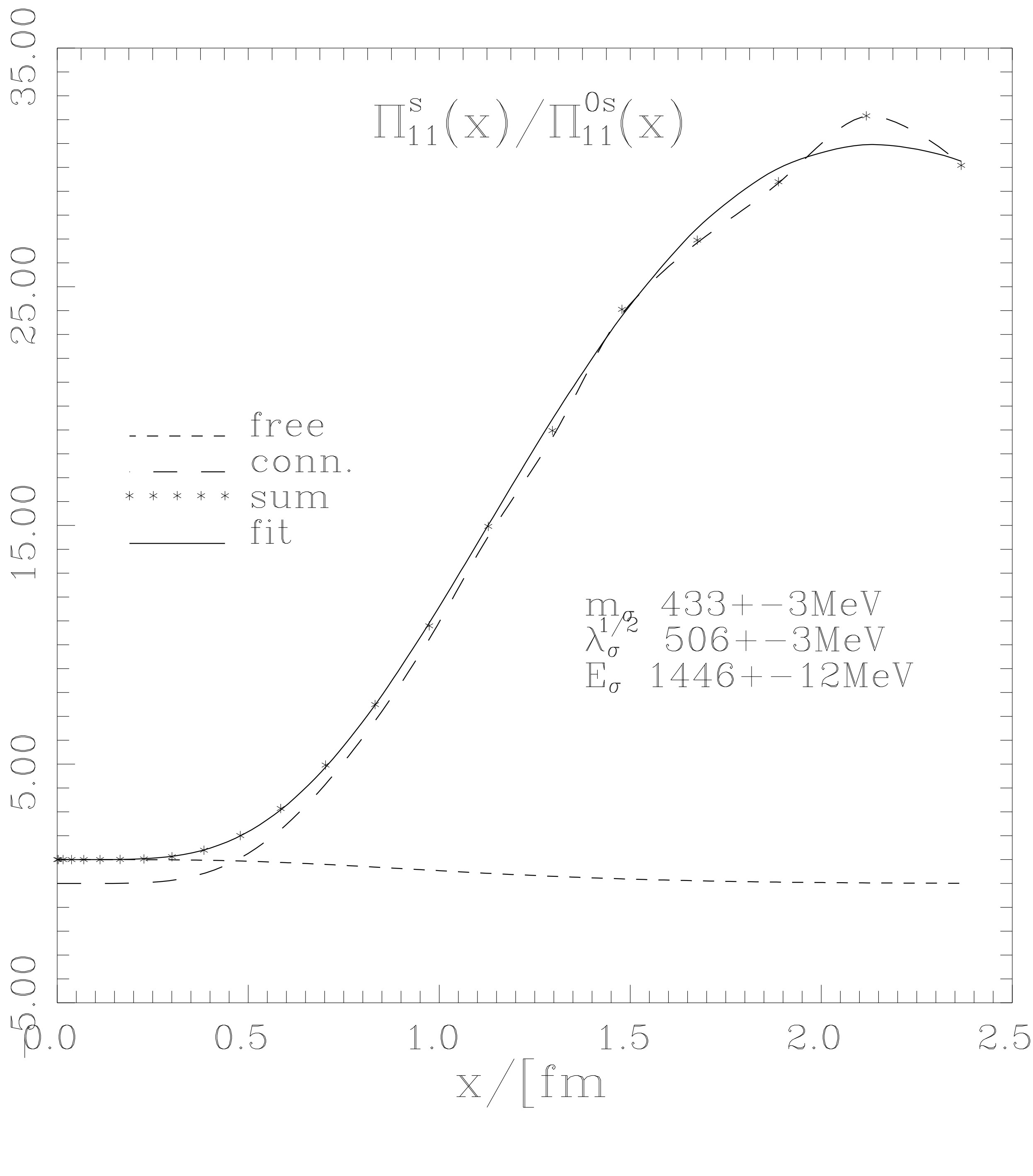


Figure B.6: *Scalar singlet correlator normalized to the free massless quark correlator. The σ mass m_σ and coupling λ_σ and the threshold E_σ are obtained from a spectral fit.*

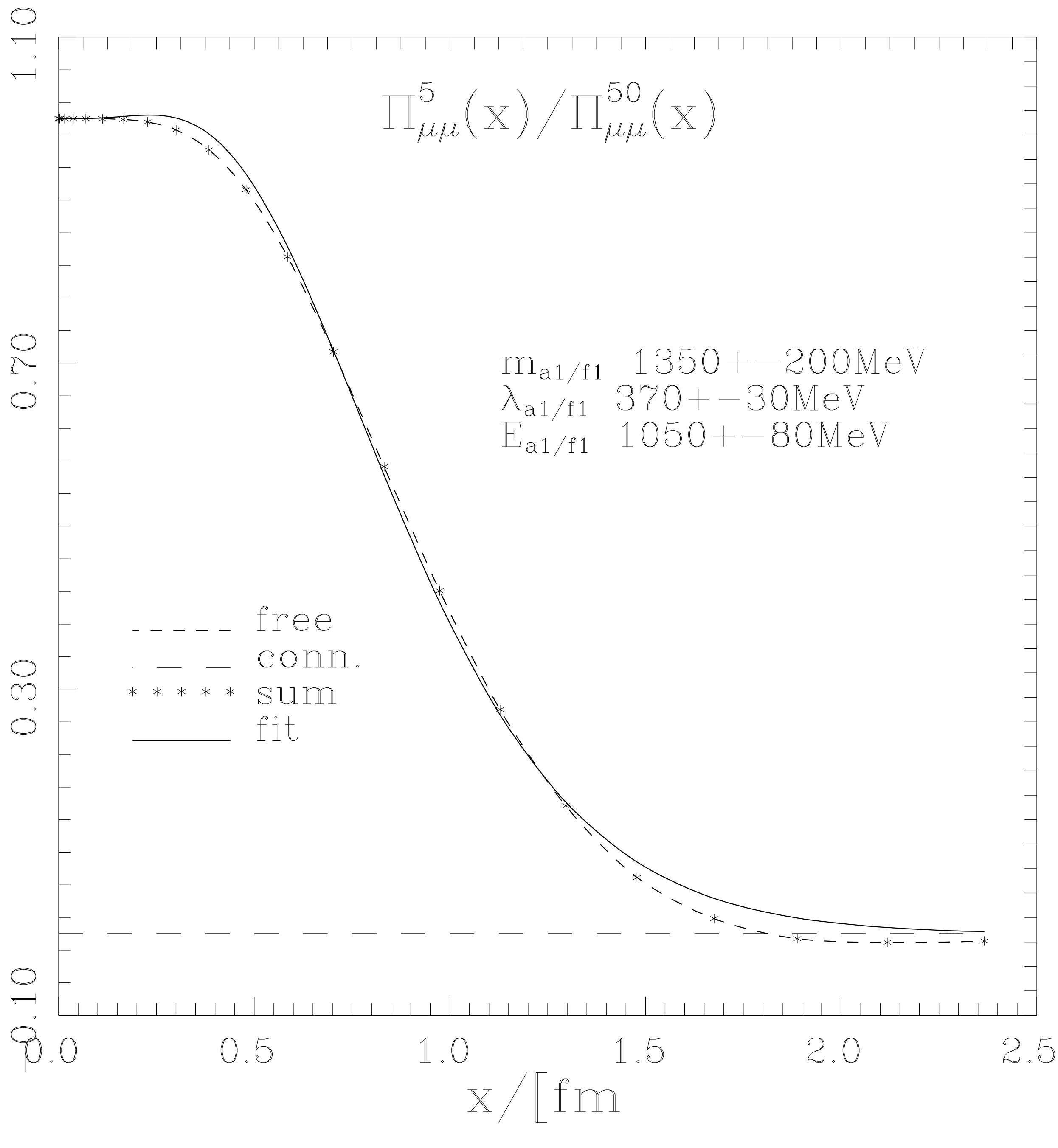


Figure B.7: *Axial vector correlator normalized to the free massless quark correlator. The triplet and singlet correlator are equal because the connected part has been neglected. The* a_1 *and* f_1 *mass, coupling and threshold are obtained from a spectral fit.*

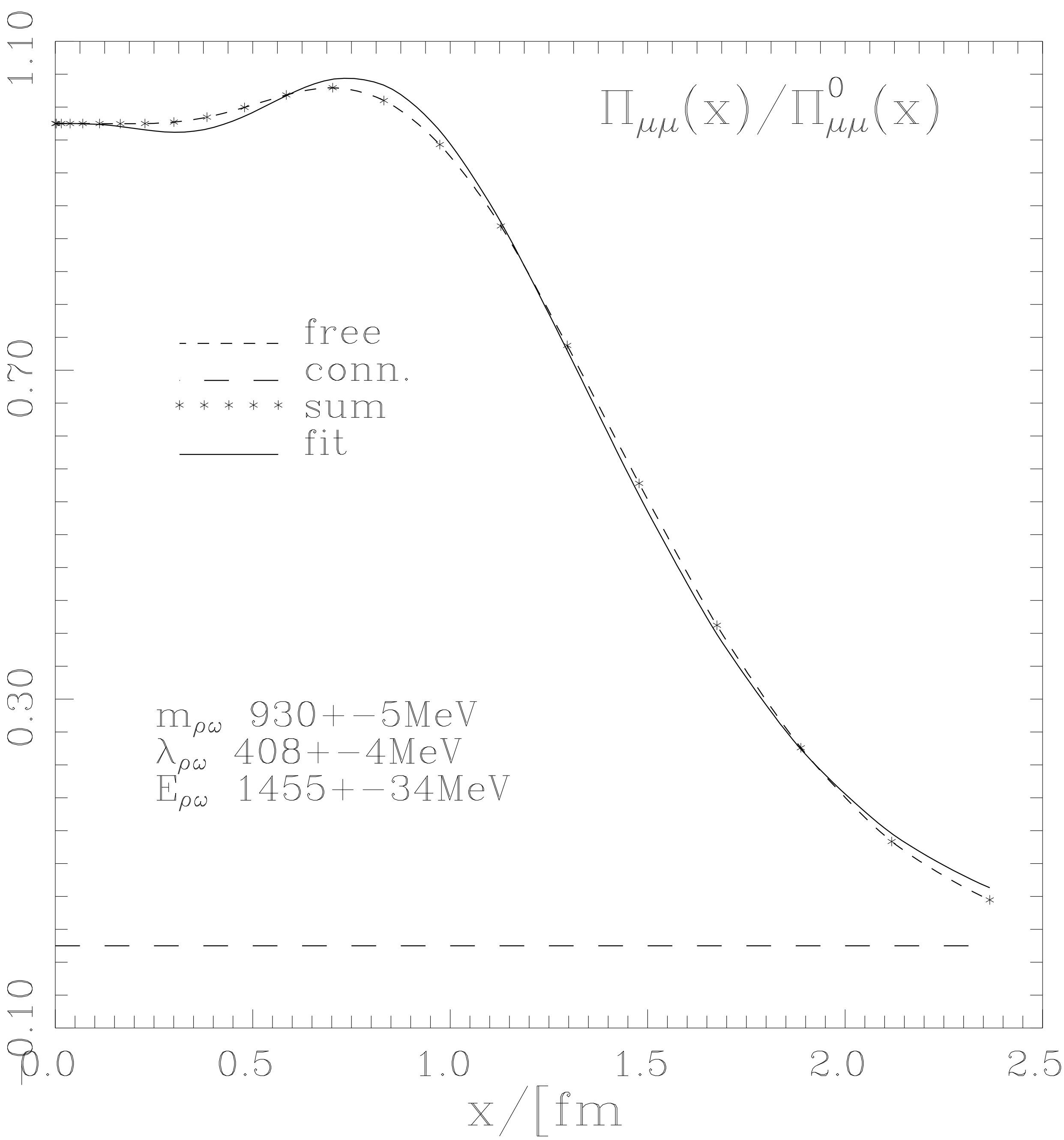


Figure B.8: *Vector correlator normalized to the free massless quark correlator. The triplet and singlet correlator are equal because the connected part is zero. The ρ and ω mass, coupling and threshold are obtained from a spectral fit.*

Bibliography

[1]

General References

[2] J. Glimm, A. Jaffe: *Quantum physics*; *New York, Springer (1981)*

[3] H. Lehman et al.: *"LSZ theory"*; *Nuovo Cim. 1 (1955) 205*

[4] J.C. Ward: *Phys.Rev. 78 (1950) 182*

[5] C.G. Callen: *Broken scale invariance in scalar field theory*; *Phys.Rev. D2 (1970) 1541*

[6] K. Symanzik: *Small distance behaviour in field theory and power counting*; *Comm.Math.Phys. 18 (1970) 227*

[7] W. Zimmermann: *Composite operators, normal products and the short distance expansion in the perturbation theory of renormalizable interactions*; *Ann.of Phys. 77 (1973) 536,570*

[8] R.P. Feynman: *Quantum mechanics and path integrals*; *Rev.Mod.Phys. 20 (1948) 367*

[9] M. Creutz: *Quarks, gluons and lattices*; *Cambridge Univ. Press (1983)*

[10] F.E. Close: *An introduction to quarks and partons*; *Academic Press (London) (1979)*

Instantons: Introductions

[11] R. Rajaraman: *Solitons and instantons*; *North-Holland (1982)*

[12] J. Zinn-Justin: *The principles of instanton calculus*; *Les Houches (1984)*

[13] M.A. Shifman et al.: *ABC of instantons*; *Fortschr.Phys. 32,11 (1984) 585*

[14] G. Callen, R. Dashen, D. Gross: *Toward a theory of strong interactions*; *Phys.Rev. D17 (1978) 2717 A theory of hadronic structure*; *Phys.Rev. D19 (1979) 1826*

Instantons: General Formulas

[15] A.A. Belavin et al.: *"Instantons in Yang-Mills-Theory"*; *Phys.Lett. B59 (1975) 85*

[16] A. Actor: *Classical solutions of SU(2) Yang-Mills theories*; *Rev.Mod.Phys. 51 (1979) 461*

[17] G. 't Hooft: *Computation of the quantum effects due to a four-dimensional pseudoparticle*; *Phys.Rev. D14 (1976) 3432*
C. Bernard: *Gauge zero modes, instanton determinants, and QCD calculations*; *Phys.Rev. D19 (1979) 3013*

[18] L.S. Brown et al.: *Propagation functions in pseudoparticle fields*; *Phys.Rev. D17 (1978) 1583*; *Massive propagators in instanton fields*; *Phys.Rev. D18 (1978) 2180*

[19] R.D. Carlitz: *Bound states from instantons*; *Phys.Rev. D17 (1978) 3225*

[20] J.M.M. Verbaarschot: *Streamlines and conformal invariance in Yang-Mills theories*; *Nucl.Phys. B362 (1991) 33*

[21] M.A. Shifman et al.: *Instanton density in a theory with massless quarks*; *Nucl.Phys. B163 (1980) 46* ; *Nucl.Phys. B191 (1981) 301*

[22] B.V. Geshkeinbein, B.L. Ioffe: *The role of instantons in generation of mesonic mass spectrum*; *Nucl.Phys. B166 (1980) 340*

[23] M. Hutter *Gauge invariant quark propagator in the instanton background*; *HEP-PH 9502361, submitted to Nucl. Phys.*

Instantons: The Liquid Model

[24] E. Ilgenfritz et al.: *Hard-core Model*; *Nucl.Phys. B184 (1981) 443*

[25] D.I. Dyakonov, V.Yu. Petrov: *Instanton-based vacuum from the feynman variational principle*; *Nucl.Phys. B245 (1984) 259*; *A theory of light quarks in the instanton vacuum*; *B272 (1985B) 457*

[26] M. Hutter *Instantons in QCD and meson correlation functions* München preprint *HEP 95-01, Submitted to Zeitschr.Phys.*

Instantons: Numerical Studies

[27] E.V. Shuryak: *The role of instantons in QCD (A1-A4)*; *Nucl.Phys B203 (1982) 93,116,140; B214 (1982) 237 Toward the quantitative theory of the instanton liquid (B1-B4)*; *Nucl.Phys B302 (1988) 559,574,599,621 Instantons in QCD (C1-C4)*; *Nucl.Phys B319 (1989) 521,541; B328 (1989) 85,102*

[28] E.V. Shuryak, J.J.M. Verbaarschot: *Quark propagation in the random instanton vacuum*; *Nucl.Phys. B410 (1993) 37 Mesonic correlation functions in the random instanton vacuum*; *Nucl.Phys. B410 (1993) 55 Baryonic correlators in the random instanton vacuum*; *Nucl.Phys. B412 (1994) 143*

[29] T. Schaefer, E.V. Shuryak: *Hadronic wave functions in the instanton liquid model*; *Phys.Rev. D50 (1994) 478*

Gluon Mass

[30] L. Scott: *Nucl. Phys. B157 (1979) 392*

[31] J.M. Cornwall: *Phys.Rev. D10 (1974) 500*

[32] K. Lane: *Phys.Rev. D10 (1974) 2605*

[33] H. Pagels: *Phys.Rev. D19 (1979) 3080*

[34] N. Bralic: *Phys.Rev. D22 (1980) 3090*

[35] P. Fishbane et al.: *Phys.Rev. D24 (1981) 2324*

[36] C.G. Callen et al.: *Phys.Lett. B66 (1977) 375*

[37] J. Ambjorn et al.: *Nucl.Phys. B152 (1979) 75, B170 (1980) 60,265*

[38] J.M. Cornwall: *Phys.Rev. D26 (1982) 1453*

[39] F. Halzen: Relating the QCD pomeron to an effective gluon mass; *preprint MAD/PH/702 (1992)*

[40] M. Hutter: *Gluon mass from instantons*; *preprint LMU-Muenchen HEP 93/18 (1993)*

Axial Anomaly, η' Mass and Proton-Spin

[41] J.S. Bell, R. Jackiw; *"Axial anomaly"*: *Nuovo.Cim. A51 (1969) 47*
S.L. Adler, W.A. Bardeen: *"Axial anomaly"*: *Phys.Rev. 182 (1969) 1517*

[42] J. Kogut, L. Susskind: *How quark confinement solves the* $\eta \rightarrow 3\pi$ *problem*; *Phys.Rev. D11 (1975) 3594*

[43] E. Witten: *Current algebra theorems for the U(1) "Goldstone boson" Nucl.Phys. B156 (1979) 269*

[44] B. Ioffe: *On the choice of quark currents in QCD sum rules for baryon masses*; *Z.Phys C18 (1983) 67*

[45] S.D. Baas, A.W. Thomas: *The EMC spin effect*; *Cavendish preprint HEP 92/5 (1992)*

[46] E. Reya: *Polarized structure functions and the spin of the proton*; *Dortmund preprint DO-TH 93/09*

[47] R.G. Roberts: *The structure of the proton*; *Cambridge (1990)*

[48] E.M. Collaboration: *An investigation of the spin structure of the proton in deep inelastic scattering of polarised muons on polarised protons*; *Nucl.Phys. B328 (1989) 1*

[49] S.M. Collaboration: *Measurement of the spin dpendent structure function* $g_1(p)$ *of the proton*; *CERN-PRE 94 (1994)*

[50] L.M. Sehgal: *"neutron* β*-decay"*; *Phys.Rev. D10 (1974) 1663*

[51] M. Bourquin et al.: *"octet hyperon* β*-decay"*; *Z.Phys. C21 (1983) 26*
J. Stern, G. Clement; *Phys. Lett. B231 (1989) 471*

[52] J. Ellis, M. Karliner: *Phys.Lett. B213 (1988) 73*

[53] S. Forte, E.V. Shuryak: *Instanton-induced supression of the singlet axial charge of the proton*; *Nucl.Phys. B357 (1991) 153*

[54] R.L. Jaffe: *The* g_1 *problem: Deep inelastic scattering and the spin of the proton*; *Phys.Lett. B193 (1987) 101*

[55] H. Fritzsch: *The spin density functions of the nucleon*; *Phys.Lett. B242 (1990) 451*
QCD anomaly and the spins inside the quarks; *Phys.Lett. B256 (1991) 75*

Operator-Product-Expansion

[56] Narison: *QCD spectral sum rules*; *World scientific 1989*

[57] M.A. Shifman et al.: *QCD and resonance physics*; *Nucl.Phys. B147 (1979) 385, 448,519*

[58] V.A. Novikov et al.: *Calculation in external fields in QCD; technical review*; *Fort.Phys. 32 (1984) 585*; *Wilson's OPE: can it fail ?*; *Nucl.Phys. B249 (1985) 445*

Miscellaneous

[59] S.V. Shabanov: *Lectures on quantization of gauge theories by the path intagral method*; *Corfu School Proceedings 1992*

[60] Schrieffer: *Theory of superconductivity*; *Benjamin (1964)*

[61] D.J. Gross, F. Wilczek: *"Asymptotic freedom"*; *Phys.Rev.Lett 30 (1973) 1343*
H.D. Politzer: *"Asymptotic freedom"*; *Phys.Rev.Lett 30 (1973) 1346*

[62] E. Witten: $1/N_c$ *expansion in QCD*; *Nucl.Phys. B160 (1979) 57*

[63] S. Weinberg: *Das* $U(1)$ *Problem Phys.Rev. D11 (1975) 3583*

[64] M.A. Shifman: *Anomalies in gauge theories*; *Phys.Rep. 209 (1991) 341*

[65] A.V. Smilga: *Vakuum fields in the schwinger model*; *Phys.Rev. D46 (1992) 5598*
H. Leutwyler, A.V. Smilga: *Spectrum of Dirac operator and role of winding number in QCD*; *Phys.Rev. D46 (1992) 5607*

[66] E.V. Shuryak: *Correlation functions in the QCD vacuum* Rev.Mod.Phys. *65 (1993) 1*

[67] K.G. Wilson: *Confinement of quarks*; *Phys.Rev. D10 (1974) 2445*

[68] J. Gasser, H. Leutwyler: *Quark masses*; *Phys.Rep. 87 (1982) 77*

[69] A.E. Dorokhov et al.: *Proton spin within nonperturbative QCD*; Int.J.Mod.Phys. *A8 (1993) 603*

Curriculum Vitae

Personal

Name:	Marcus Hutter
Adress:	Josephsburgstr. 59 D-81673 Munich
Birth:	14. April 1967 in Munich

Schooling

Sep.1973 - Jul.1977	Primary School in Munich and USA
Sep.1977 - Mai.1987	High School in Munich and Markt-Schwaben Majors in Math & Physics
20.Mai.1987	Degree: Abitur

Degree in Computer Sciences

Nov.1987 - Mai.1992	Computer Science with minors in Math at the Technical University Munich. Main area of study: Artificial Intelligence, Neuronal Nets, Genetic Algorithms
Master Thesis:	Implementation of a Classfier System
26.Mai.1992	Degree: Masters (Diplom)

Degree in Physics

Nov.1989 - Jul.1993	General Physics at the Technical University Munich
Nov.1992 - Jan.1996	Ph.D. thesis in theoretical particle physics. Supervisor Prof. H. Fritzsch. Topic: Instantons in QCD